arXiv:0706.0268v1 [math-ph] 2 Jun 2007

Forward and backward time observables for quantum evolution and quantum stochastic processes -I: The time observables

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Abstract

Given a Hamiltonian $\mathbf{H}$ on a Hilbert space $\mathcal{H}$ it is shown that, under the assumption that $\sigma(\mathbf{H}) = \sigma_{ac}(\mathbf{H}) = \mathbb{R}^+$, there exist unique positive operators $\mathbf{T}_F$ and $\mathbf{T}_B$ registering the Schrödinger time evolution generated by $\mathbf{H}$ in the forward (future) direction and backward (past) direction respectively. These operators may be considered as time observables for the quantum evolution. Moreover, it is shown that the same operators may serve as time observables in the construction of quantum stochastic differential equations and quantum stochastic processes in the framework of the Hudson-Parthasarathy quantum stochastic calculus. The basic mechanism enabling for the definition of the time observables originates from the recently developed semigroup decomposition formalism used in the description of the time evolution of resonances in quantum mechanical scattering problems.

1 Introduction

The recently developed semigroup decomposition formalism for the description of the time evolution of quantum mechanical resonances [28, 29, 27] utilizes two central ingredients, namely the Sz.-Nagy-Foias theory of contraction operators and strongly contractive semigroups on Hilbert space [32] and the contractive nesting of Hilbert spaces, i.e., the embedding of one Hilbert space into another via a contractive quasi-affine transformation [13], in order to decompose the time evolution of resonances in standard, non-relativistic, quantum mechanical scattering problems into a sum of a semigroup part and a non-semigroup part. In this decomposition the semigroup part, given in terms of a Lax-Phillips type semigroup (see for example [27] for the terminology used here), is the resonance term and the non-semigroup part is called the background term. The complex eigenvalues of the generator of the semigroup, providing the typical exponential decay behaviour of the resonance part, are associated with resonance poles of the scattering matrix. In fact, under appropriate conditions, the scattering matrix can be factored and the rational part of this factorization corresponds to the characteristic function (see for example [32]) of the cogenerator of the (adjoint of) the resonance part semigroup. By a theorem of Sz.-Nagy and Foias ([32], Chap. VI, Sec. 4) the characteristic

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function determines uniquely the spectrum of the generator of the resonant part Lax-Phillips type semigroup. The close relation between resonance poles of the S-matrix and the time evolution of a resonance is thus clearly exhibited.

As mentioned above, the semigroup decomposition formalism uses the Sz.-Nagy-Foias theory in order to extract the semigroup part of the evolution of a resonance. Specifically, use is made of a Hardy space functional model for the $C_{\cdot 0}$ class contractive semigroup (a Lax-Phillips type semigroup in the terminology used above) corresponding to the exponentially decaying resonance part of the evolution. This functional model is associated with the construction of isometric dilations of $C_{\cdot 0}$ class semigroups [32, 18, 14, 20, 34, 35] (see also [28] for a short review of the mathematical structures involved). A central ingredient of this functional model is a particular semigroup on Hardy space which we will presently define.

Denote by $\mathbb{C}^+$ the upper half of the complex plane and let $\mathcal{H}^2_{\mathcal{N}}(\mathbb{C}^+)$ be the Hardy space of vector valued functions analytic in the upper half-plane and taking values in a separable Hilbert space $\mathcal{N}$. Similarly, $\mathcal{H}^2_{\mathcal{N}}(\mathbb{C}^-)$ denotes the Hardy space of $\mathcal{N}$ valued functions analytic in the lower half-plane $\mathbb{C}^-$. The set of boundary values on $\mathbb{R}$ of functions in $\mathcal{H}^2_{\mathcal{N}}(\mathbb{C}^+)$ is a Hilbert space isomorphic to $\mathcal{H}^2_{\mathcal{N}}(\mathbb{C}^+)$ which we denote by $\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})$. In a similar manner $\mathcal{H}^-_{\mathcal{N}}(\mathbb{R})$, the space of boundary values of functions in $\mathcal{H}^2_{\mathcal{N}}(\mathbb{C}^-)$, is isomorphic to $\mathcal{H}^2_{\mathcal{N}}(\mathbb{C}^-)$. Throughout the present paper we mostly deal with $\dim\mathcal{N}=1$, i.e., the case scalar valued functions. In this case we denote by $\mathcal{H}^2(\mathbb{C}^+)$ and $\mathcal{H}^2(\mathbb{C}^-)$ the Hardy spaces of the upper half-plane and lower half-plane respectively. The spaces of boundary value functions on the real axis are then denoted by $\mathcal{H}^+(\mathbb{R})$ and $\mathcal{H}^-(\mathbb{R})$. Define a family $\{u(t)\}_{t\in\mathbb{R}}$ of unitary, multiplicative operators $u(t): L^2_{\mathcal{N}}(\mathbb{R})\mapsto L^2_{\mathcal{N}}(\mathbb{R})$ by

$$[u(t)f](\sigma)=e^{-i\sigma t}f(\sigma),\quad f\in L^2_{\mathcal{N}}(\mathbb{R}),\quad \sigma\in\mathbb{R}\,, \tag{1}$$

The family $\{u(t)\}_{t\in\mathbb{R}}$ forms a one parameter group of multiplicative operators on $L^2_{\mathcal{N}}(\mathbb{R})$. Let P_+ be the orthogonal projection of $L^2_{\mathcal{N}}(\mathbb{R})$ on $\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})$. A *Toeplitz operator* with symbol $u(t)$ [21, 22, 33] is an operator $T^+_u(t):\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})\mapsto\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})$ defined by

$$T^+_u(t)f:=P_+u(t)f,\qquad f\in\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})\,. \tag{2}$$

The set $\{T^+_u(t)\}_{t\in\mathbb{R}^+}$ forms a strongly continuous, contractive, one parameter semigroup on $\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})$ satisfying

$$s-\lim_{t\to\infty}T^+_u(t)=0\,.$$

Moreover, we have

$$\|T^+_u(t_2)f\|\le\|T^+_u(t_1)f\|,\qquad t_2>t_1,\ f\in\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})\,.$$

The functional model providing the semigroup evolution of the resonance term in the semigroup decomposition formalism is obtained by the compression of the semigroup $\{T^+_u(t)\}_{t\in\mathbb{R}^+}$ to an invariant subspace $\hat{\mathcal{K}}\subset\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})$ given by

$$\hat{\mathcal{K}}=\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})\ominus\Theta_T(\cdot)\mathcal{H}^+_{\mathcal{N}}(\mathbb{R}) \tag{3}$$

where $\Theta_T(\cdot):\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})\mapsto\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})$ is an *inner function* [33, 21] (also [15, 9] for the scalar case) for $\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})$. In the semigroup decomposition formalism $\Theta_T(\cdot)$ is associated with the scattering matrix. In fact, it corresponds to the rational factor in the factorization of the S-matrix mentioned above.

Observe that the semigroups above are defined in terms of abstract function spaces i.e., Hardy spaces and certain subspaces of Hardy spaces. A natural question is how are these semigroups linked to the physical Hilbert space and physical evolution of a given quantum mechanical system. In the semigroup decomposition formalism this is the role of the quasi-affine transform mentioned above. We first recall the definition of a quasi-affine map (in Definition 1 and through the rest of the paper $\mathcal{B}(\mathcal{H})$ denotes the space of bounded linear operators on $\mathcal{H}$)

Definition 1 (quasi-affine map) *A quasi-affine map from a Hilbert space $\mathcal{H}_1$ into a Hilbert space $\mathcal{H}_0$ is a linear, injective, continuous mapping of $\mathcal{H}_1$ into a dense linear manifold in $\mathcal{H}_0$. If $\mathbf{A} \in \mathcal{B}(\mathcal{H}_1)$ and $\mathbf{B} \in \mathcal{B}(\mathcal{H}_0)$ then $\mathbf{A}$ is a quasi-affine transform of $\mathbf{B}$ if there is a quasi-affine map $\theta : \mathcal{H}_1 \mapsto \mathcal{H}_0$ such that $\theta\mathbf{A} = \mathbf{B}\theta$.*

If $\theta : \mathcal{H}_1 \mapsto \mathcal{H}_0$ is a quasi-affine mapping then $\theta^* : \mathcal{H}_0 \mapsto \mathcal{H}_1$ is also quasi-affine, that is θ^* is one to one, continuous and its range is dense in $\mathcal{H}_1$. Moreover, if $\theta_1 : \mathcal{H}_0 \mapsto \mathcal{H}_1$ is quasi-affine and $\theta_2 : \mathcal{H}_1 \mapsto \mathcal{H}_2$ is quasi-affine then $\theta_2\theta_1 : \mathcal{H}_0 \mapsto \mathcal{H}_2$ is quasi-affine [32].

Consider a seperable Hilbert space $\mathcal{H}$ and a one parameter unitary evolution group $\{\mathbf{U}(t)\}_{t\in\mathbb{R}}$ on $\mathcal{H}$ generated by a self-adjoint Hamiltonian $\mathbf{H}$. We will assume that the spectrum of $\mathbf{H}$ satisfies $\sigma(\mathbf{H}) = \sigma_{ac}(\mathbf{H}) = ess\,supp\,\sigma_{ac}(\mathbf{H}) = \mathbb{R}^+$. For simplicity we will assume furthermore that the multiplicity of $\sigma(\mathbf{H})$ is one. By a slight variation of a fundamental theorem proved in reference [28] one can then prove the existence of a mapping $\Omega_f : \mathcal{H} \mapsto \mathcal{H}^+(\mathbb{R})$ such that

(i) Ω_f is a contractive quasi-affine mapping of $\mathcal{H}$ into $\mathcal{H}^+(\mathbb{R})$.

(ii) For $t \geq 0$, $\mathbf{U}(t)$ is a quasi-affine transform of the Toeplitz operator $T_u^+(t)$. For every $g \in \mathcal{H}$ we have
$$\Omega_f \mathbf{U}(t) g = T_u^+(t)\Omega_f g, \qquad t \geq 0\,. \tag{4}$$

(here the subscript f in Ω_f designates forward time evolution). We note that in the case of the semigroup decomposition one constructs two different such quasi-affine transformations, denoted $\hat{\Omega}_\pm$ and corresponding respectively to the two Møller wave operators for a given scattering problem. One then defines what is called the nested S-matrix $S_{nest} := \hat{\Omega}_+\hat{\Omega}_-^{-1}$ (see appropriate definitions in [29]).

By the remarks following the definition above of a quasi-affine mapping, if $\Omega_f : \mathcal{H} \mapsto \mathcal{H}^+(\mathbb{R})$ is quasi-affine then $\Omega_f^* : \mathcal{H}^+(\mathbb{R}) \mapsto \mathcal{H}$ is quasi-affine with range dense in $\mathcal{H}$. Denoting $\mathbf{T}_F^{-1} := \Omega_f^*\Omega_f$ we find that $\mathbf{T}_F^{-1} : \mathcal{H} \mapsto \mathcal{H}$ is a quasi-affine mapping with range dense in $\mathcal{H}$. Setting $\Sigma_{\Omega_f} := \mathrm{Ran}\,(\mathbf{T}_F^{-1})$ one can show that $\mathbf{T}_F^{-1}$ is a positive, bounded, self-adjoint operator with a self-adjoint inverse $\mathbf{T}_F : \Sigma_{\Omega_f} \mapsto \mathcal{H}$ (see Section 2). We have

Definition 2 (forward time observables) *Let $\Omega_f : \mathcal{H} \mapsto \mathcal{H}^+(\mathbb{R})$ be a mapping satisfying (i)-(ii) above, and let $\Sigma_{\Omega_f} = \Omega_f^*\Omega_f\,\mathcal{H}$ and $\hat{\Sigma}_{\Omega_f} = \Omega_f\Omega_f^*\mathcal{H}^+(\mathbb{R})$. Then the operator $\hat{T}_F : \hat{\Sigma}_{\Omega_f} \mapsto \mathcal{H}^+(\mathbb{R})$ defined by*
$$\hat{T}_F := (\Omega_f^*)^{-1}\Omega_f^{-1}$$
will be called the Hardy space forward time observable and the operator $\mathbf{T}_F : \Sigma_{\Omega_f} \mapsto \mathcal{H}$ defined by
$$\mathbf{T}_F := \Omega_f^{-1}(\Omega_f^*)^{-1}$$
will be called the physical forward time observable. □

It is proved in Section 2 that $\inf \sigma(\mathbf{T}_F) = 1$ where $\sigma(\mathbf{T}_F)$ denotes the spectrum of $\mathbf{T}_F$. An extended version of the following theorem is also proved in Section 2:

Theorem 1 *Let $\xi_{\mathcal{H}}$ be the spectral measure (i.e. spectral projection valued measure) of $\mathbf{T}_F$ and let $a > 1$. Then for any $g \in \mathcal{H}$ satisfying $\xi_{\mathcal{H}}([1, a))g = g$ there exists $\tau > 0$ such that $\xi_{\mathcal{H}}([a, \infty))\mathbf{U}(t)g \neq \{0\}$ for all $t > \tau$ and, moreover, $\lim_{t\to\infty} \|\xi_{\mathcal{H}}([1, a))\mathbf{U}(t)g\|_{\mathcal{H}} = 0$.*

□

Theorem 1 states that if $g \in \mathcal{H}$ is compactly supported on the spectrum of $\mathbf{T}_F$ then the evolved state $g(t) = \mathbf{U}(t)g$ in the forward direction of time (i.e., for $t \geq 0$) must "go up" on the spectrum of $\mathbf{T}_F$ as time increases. Therefore, *a priori* $\mathbf{T}_F$ may be regarded as an observable registering the flow of time in the system in the future direction. This observation provides the motivation for the terminology used for $\mathbf{T}_F$ in the definition above. Note that $\mathbf{T}_F$ is not a time observable in the sense of a Mackey imprimitivity system [19] (such a time observable does not exist for problems where the generator of evolution is semibounded).

Although formulated completely within the framework of standard quantum mechanics, where the evolution of a system is given in terms of a one parameter group $\{\mathbf{U}(t)\}_{t\in\mathbb{R}}$ generated by a self-adjoint Hamiltonian $\mathbf{H}$, the semigroup decomposition formalism holds its origines in a particular line of investigation associated with recent efforts to understand irreversible quantum evolution (in this respect see [23, 10, 16, 28, 31, 1, 2]). The time asymmetry built into this framework is clearly exhibited in property (ii) above where the intertwining of $\mathbf{U}(t)$ and $T_u^+(t)$ through Ω_f is valid only for $t \geq 0$. In fact, one may apply the semigroup decomposition also in the backward direction of time using the lower half-plane Hardy space $\mathcal{H}^-(\mathbb{R})$ and a different quasi-affine mapping. Denoting by P_- the orthogonal projection of $L^2(\mathbb{R})$ on $\mathcal{H}^-(\mathbb{R})$ we consider in $\mathcal{H}^-(\mathbb{R})$ the family of operators $T_u^-(t) : \mathcal{H}^-(\mathbb{R}) \mapsto \mathcal{H}^-(\mathbb{R})$ defined by

$$T_u^-(t)f := P_- u(t)f, \qquad f \in \mathcal{H}^-(\mathbb{R})\,.$$

The set $\{T_u^-(-t)\}_{t\in\mathbb{R}^+}$ forms a strongly continuous, contractive, one parameter semigroup on $\mathcal{H}^-(\mathbb{R})$. Then, under the same assumptions leading to (i) and (ii) above, there exists a transformation $\Omega_b : \mathcal{H} \mapsto \mathcal{H}^-(\mathbb{R})$ with the properties

(i') Ω_b is a contractive quasi-affine mapping of $\mathcal{H}$ into $\mathcal{H}^-(\mathbb{R})$.

(ii') For $t \leq 0$, $\mathbf{U}(t)$ is a quasi-affine transform of $T_u^-(t)$. For every $f \in \mathcal{H}$ we have

$$\Omega_b \mathbf{U}(t)f = T_u^-(t)\Omega_b f, \qquad t \leq 0\,. \tag{5}$$

(here the subscript b in Ω_b designates backward time evolution). Using Eq. (4) and Eq. (5) in the semigroup decomposition formalism the description of the evolution of a system may be splits into two different representations, one in the forward direction and one in the backward direction, according to the different semigroup acting in each of them. The structure thus obtained resembles, and is in fact closely related to, the use of Hardy spaces in a rigged Hilbert space formulation of the problem of resonances [3, 4, 5, 6, 11, 12] where the evolution also splits into a semigroup acting in the forward direction and a differnt semigroup acting in the backward direction.

Associated with the quasi-affine mapping Ω_b are the Hardy space and physical space backward time observables

Definition 3 (backward time observables) *Let $\Omega_b : \mathcal{H} \mapsto \mathcal{H}^-(\mathbb{R})$ be a map satisfying (i')-(ii') above and let $\Sigma_{\Omega_b} := \Omega_b^* \Omega_b \mathcal{H}$ and $\hat{\Sigma}_{\Omega_b} := \Omega_b \Omega_b^* \mathcal{H}^-(\mathbb{R})$. Then the operator $\hat{T}_B : \hat{\Sigma}_{\Omega_b} \mapsto \mathcal{H}^-(\mathbb{R})$ defined by*

$$\hat{T}_B := (\Omega_b^*)^{-1} \Omega_b^{-1}$$

will be called the Hardy space backward time observable and the operator $\mathbf{T}_B : \Sigma_{\Omega_b} \mapsto \mathcal{H}$ defined by

$$\mathbf{T}_B := \Omega_b^{-1} (\Omega_b^*)^{-1}$$

will be called the physical backward time observable. □

A Theorem analogous to Theorem 1 holds for $\mathbf{T}_B$ for negative times. The reader is referred to Theorem 2 in Section 2.

Considering the existence of distinct time observables for forward and backward evolution and the *a priori* time asymmetry existing in the formalism from which they arise, one may naturally ask whether these operators can be used as time observables not only for a quantum system undergoing Schrödinger evolution but more generally for quantum irreversible processes such as quantum stochastic processes. Another question is whether relations similar to those in (ii) and (ii') above between evolution in Hardy space and evolution in physical space are again exhibited in this more general context. In Section 3 below we consider the time observables $\mathbf{T}_F$, $\hat{T}_F$ in the framework of the Hudson-Pharthasarathy (HP) quantum stochastic calculus [17, 24, 25] and show that indeed these operators can be used as time observables for quantum stochastic processes and that the stochastic processes defined with respect to the (second quantisation of) Hardy space can be mapped to corresponding stochastic processes defined with respect to the (second quantisation of) physical space through a mapping associated with the quasi-affine map $\Omega_f : \mathcal{H} \mapsto \mathcal{H}^+(\mathbb{R})$.

Some remarks concerning notation: With the exception of the identity, operators acting in $\mathcal{H}$ are denoted below by capital bold face letters. Thus $\mathbf{H}$, $\mathbf{K}$, $\mathbf{T}_F$ etc. are all operators in $\mathcal{H}$. Operators acting in the Hardy spaces $\mathcal{H}^+(\mathbb{R})$ and $\mathcal{H}^-(\mathbb{R})$ are denoted by a hat. Thus $\hat{H}$, $\hat{K}$, $\hat{T}_F$ etc. are operators in Hardy spaces. Unless otherwise specifically stated in the text the identity operators in the various Hilbert spaces are generically denote simply by 1 with the exact meaning implied by the particular context. The Borel σ-algebra of $\mathbb{R}^+$ is denoted by $\mathfrak{B}^+$ and the set of all projection operators in a Hilbert space $\mathcal{H}$ is denoted $\mathcal{P}(\mathcal{H})$ (thus, for example, to a positive self-adjoint operator $\mathbf{A}$ there corresponds by the spectral theorem a spectral projection valued measure, say ξ, such that $\xi : \mathfrak{B}^+ \mapsto \mathcal{P}(\mathcal{H})$). In addition the norm in the various Hilbert spaces is denoted by the appropriate subscript; in particular $\|\cdot\|_{\mathcal{H}^+}$ denotes the norm in $\mathcal{H}^+(\mathbb{R})$, $\|\cdot\|_{\mathcal{H}^-}$ denotes the norm in $\mathcal{H}^-(\mathbb{R})$ etc. .

The rest of this paper is organized as follows: In Subsection 2.1 an exact definition of forward and backward time observables is given followed by a discussion of several of their basic properties such as domain of definition, positivity, self-adjointness etc. . Subsequently, Theorem 2 in the same subsection establishes the motivation for the terminology "physical time observables" applied to the operators $\mathbf{T}_F$ and $\mathbf{T}_B$. This theorem is of central importance in the context of the present work. Section 2 concludes in Subsection 2.2 with a discussion of the mathematical structure enabling the existence of time observables. The application of the operators $\mathbf{T}_F$, $\hat{T}_F$ (and $\mathbf{T}_B$, $\hat{T}_B$) as time observables for quantum stochastic processes is discussed in Section 3. The goal is to find the analogue of the fundamental intertwining relation in Eq. (4) in the context of quantum stochastic processes. This is done in two steps; first Subsection 3.1 deals with the mapping of basic creation, annihilation and conservation

processes defined with respect to $\mathbf{T}_F$ into the corresponding processes defined with respect to $\hat{T}_F$, then Subsection 3.2 contains an application of the mapping defined in the previous subsection to an important class of quantum stochastic differential equations and an analogue of Eq. (4) is obtained for this class. Section 4 contains a summary of results and open questions.

2 The time observables

2.1 Definition and basic properties

We first remark that statements concerning forward time observables $\mathbf{T}_F$ and $\hat{T}_F$ and backward time observables $\mathbf{T}_B$ and $\hat{T}_B$ are proved essentially using the same methods with obvious necessary changes. Therefore for the sake of completeness theorems are stated with specific reference to forward and backward time observables whereas detailed proofs are given for the forward time observables with an indication of the necessary replacements pertaining to backward time observables.

As in Section 1 above let $\mathbf{T}_F^{-1} : \mathcal{H} \mapsto \mathcal{H}$ be defined by $\mathbf{T}_F^{-1} := \Omega_f^* \Omega_f$ and $\hat{T}_F^{-1} : \mathcal{H}^+(\mathbb{R}) \mapsto \mathcal{H}^+(\mathbb{R})$ be defined by $\hat{T}_F^{-1} := \Omega_f\, \Omega_f^*$. Let $\Sigma_{\Omega_f} = \operatorname{Ran} \mathbf{T}_F^{-1} = \Omega_f^* \Omega_f\, \mathcal{H}$ and $\hat{\Sigma}_{\Omega_f} = \operatorname{Ran} \hat{T}_F^{-1} = \Omega_f\, \Omega_f^*\, \mathcal{H}^+(\mathbb{R})$. Let $\mathbf{T}_B^{-1} : \mathcal{H} \mapsto \mathcal{H}$ be defined by $\mathbf{T}_B^{-1} := \Omega_b^* \Omega_b$ and $\hat{T}_B^{-1} : \mathcal{H}^-(\mathbb{R}) \mapsto \mathcal{H}^-(\mathbb{R})$ be defined by $\hat{T}_B^{-1} := \Omega_b \Omega_b^*$. Let $\Sigma_{\Omega_b} = \operatorname{Ran} \mathbf{T}_B^{-1} = \Omega_b^* \Omega_b \mathcal{H}$ and $\hat{\Sigma}_{\Omega_b} = \operatorname{Ran} \hat{T}_B^{-1} = \Omega_b \Omega_b^*\, \mathcal{H}^-(\mathbb{R})$. We have

Proposition 1 *The operator $\mathbf{T}_F : \mathcal{D}(\mathbf{T}_f) \mapsto \mathcal{H}$ given by $\mathbf{T}_F := (\mathbf{T}_F^{-1})^{-1} = \Omega_f^{-1}(\Omega_f^*)^{-1}$ is a positive, unbounded self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(\mathbf{T}_F) = \Sigma_{\Omega_f}$. Similarly, the operator $\hat{T}_F : \mathcal{D}(\hat{T}_F) \mapsto \mathcal{H}^+(\mathbb{R})$ given by $\hat{T}_F := (\hat{T}_F^{-1})^{-1} = (\Omega_f^*)^{-1}\Omega_f^{-1}$ is a positive, unbounded self-adjoint operator in $\mathcal{H}^+(\mathbb{R})$ with domain $\mathcal{D}(\hat{T}_f) = \hat{\Sigma}_{\Omega_f}$. The spectrums $\sigma(\mathbf{T}_F)$ and $\sigma(\hat{T}_F)$ satisfy $\sigma(\mathbf{T}_F) = \sigma(\hat{T}_F)$. If $\xi_{\mathcal{H}^+} : \mathfrak{B}^+ \mapsto \mathcal{P}(\mathcal{H}^+(\mathbb{R}))$ is the spectral measure (i.e., spectral projection valued measure) of $\hat{T}_F$ and $\xi_{\mathcal{H}} : \mathfrak{B}^+ \mapsto \mathcal{P}(\mathcal{H})$ is the spectral measure of $\mathbf{T}_F$ then, for any set $E \in \mathfrak{B}^+$ we have*

$$\xi_{\mathcal{H}}(E)g = \Omega_f^{-1}\xi_{\mathcal{H}^+}(E)\Omega_f g, \qquad g \in Ran\,\Omega_f^*\,, \tag{6}$$

$$\xi_{\mathcal{H}^+}(E)g = (\Omega_f^*)^{-1}\xi_{\mathcal{H}}(E)\Omega_f^* g, \qquad g \in Ran\,\Omega_f\,, \tag{7}$$

so that

$$\xi_{\mathcal{H}}(E) = \overline{\Omega_f^{-1}\xi_{\mathcal{H}^+}(E)\Omega_f}, \qquad \xi_{\mathcal{H}^+}(E) = \overline{(\Omega_f^*)^{-1}\xi_{\mathcal{H}}(E)\Omega_f^*}\,.$$

The operator $\mathbf{T}_B : \mathcal{D}(\mathbf{T}_B) \mapsto \mathcal{H}$ given by $\mathbf{T}_B := (\mathbf{T}_B^{-1})^{-1} = \Omega_b^{-1}(\Omega_b^*)^{-1}$ is a positive, unbounded self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(\mathbf{T}_B) = \Sigma_{\Omega_b}$. Similarly, the operator $\hat{T}_B : \mathcal{D}(\hat{T}_F) \mapsto \mathcal{H}^-(\mathbb{R})$ given by $\hat{T}_B := (\hat{T}_B^{-1})^{-1} = (\Omega_b^*)^{-1}\Omega_b^{-1}$ is a positive, unbounded self-adjoint operator in $\mathcal{H}^-(\mathbb{R})$ with domain $\mathcal{D}(\hat{T}_F) = \hat{\Sigma}_{\Omega_b}$. The spectrums $\sigma(\mathbf{T}_B)$ and $\sigma(\hat{T}_B)$ satisfy $\sigma(\mathbf{T}_B) = \sigma(\hat{T}_B)$. If $\zeta_{\mathcal{H}^-} : \mathfrak{B}^+ \mapsto \mathcal{P}(\mathcal{H}^-(\mathbb{R}))$ is the spectral measure of $\hat{T}_B$ then, for any set $E \in \mathfrak{B}^+$ we have

$$\zeta_{\mathcal{H}}(E)g = \Omega_b^{-1}\zeta_{\mathcal{H}^-}(E)\Omega_b g, \qquad g \in Ran\,\Omega_b^*\,, \tag{8}$$

$$\zeta_{\mathcal{H}^-}(E)g = (\Omega_b^*)^{-1}\zeta_{\mathcal{H}}(E)\Omega_b^* g, \qquad g \in Ran\,\Omega_b\,, \tag{9}$$

so that

$$\zeta_{\mathcal{H}}(E) = \overline{\Omega_b^{-1}\zeta_{\mathcal{H}^-}(E)\Omega_b}, \qquad \zeta_{\mathcal{H}^-}(E) = \overline{(\Omega_b^*)^{-1}\zeta_{\mathcal{H}}(E)\Omega_b^*}\,.$$

□

Proof of Proposition 1:

Clearly the operators $\mathbf{T}_F^{-1}$ and $\hat{T}_F^{-1}$ are positive and symmetric. Since both Ω_f and Ω_f^* are quasi-affine maps we have $\operatorname{Ker}\mathbf{T}_F = \{0\}$, $\operatorname{Ker}\hat{T}_F = \{0\}$ and, since both are also contractive we have $\|\mathbf{T}_F\| \leq 1$, $\|\hat{T}_F\| \leq 1$ so that, in particular, $\operatorname{Dom}\mathbf{T}_F^{-1} = \mathcal{H}$ and $\operatorname{Dom}\hat{T}_F^{-1} = \mathcal{H}^+(\mathbb{R})$. We conclude that T_F^{-1} and $\hat{T}_F^{-1}$ are self-adjoint. Moreover, again by the fact that both Ω_f and Ω_f^* are quasi-affine, we have that $\operatorname{Ran}\mathbf{T}_F^{-1} = \Sigma_{\Omega_f} \subset \mathcal{H}$ is dense in $\mathcal{H}$ and $\operatorname{Ran}\hat{T}_F^{-1} = \hat{\Sigma}_{\Omega_f} \subset \mathcal{H}^+(\mathbb{R})$ is dense in $\mathcal{H}^+(\mathbb{R})$. Therefore, $\mathbf{T}_F^{-1}$ and $\hat{T}_F^{-1}$ are invertible on a dense domain and so (see for example [26]) $\mathbf{T}_F = (\mathbf{T}_F^{-1})^{-1}$ and $\hat{T}_F = (\hat{T}_F^{-1})^{-1}$ are self-adjoint with domains $\operatorname{Dom}\mathbf{T}_F = \Sigma_{\Omega_f}$ and $\operatorname{Dom}\hat{T}_F = \hat{\Sigma}_{\Omega_f}$ respectively.

The operators $\mathbf{T}_F$ and $\hat{T}_F$ cannot be extended to bounded operators. consider the two maps $\Omega_f : \mathcal{H} \mapsto \mathcal{H}^+(\mathbb{R})$ and $\Omega_f^* : \mathcal{H}^+(\mathbb{R}) \mapsto \mathcal{H}$. Since both maps are quasi-affine we know that $\operatorname{Ran}\Omega_f$ is dense in $\mathcal{H}^+(\mathbb{R})$ and $\operatorname{Ran}\Omega_f^*$ is dense in $\mathcal{H}$. By the injective property of quasi-affine maps the two maps are also invertible on their range. Furthermore, we have (see [26])

$$(\Omega_f^{-1})^* = (\Omega_f^*)^{-1}\,. \tag{10}$$

It follows from Eq. (10) that Ω_f^{-1} and $(\Omega_f^*)^{-1}$ cannot be bounded. For $(\Omega_f^*)^{-1} : \Omega_f^*\mathcal{H}^+(\mathbb{R}) \mapsto \mathcal{H}$ is onto and if $(\Omega_f^*)^{-1}$ is bounded on $\operatorname{Ran}\Omega_f^*$ we can extend it uniquely to a bounded map defined on all of $\mathcal{H}$ which we again denote by $(\Omega_f^*)^{-1}$. Then this extended map must have a non-trivial kernel. However, assuming that $f \in \operatorname{Ker}(\Omega_f^*)^{-1}$, for arbitrary $g \in \mathcal{H}$ we have

$$0 = ((\Omega_f^*)^{-1}f, \Omega_f g)_{\mathcal{H}^+(\mathbb{R})} = ((\Omega_f^{-1})^* f, \Omega_f g)_{\mathcal{H}^+(\mathbb{R})} = (f, g)_{\mathcal{H}}$$

which is impossible unless $f = 0$. We conclude that $(\Omega_f^*)^{-1}$ is unbounded. Similarly, $\Omega_f^{-1} : \Omega_f\mathcal{H} \mapsto \mathcal{H}^+(\mathbb{R})$ is onto and if it can be extended to a bounded map, again denoted by Ω_f^{-1}, defined on all of $\mathcal{H}^+(\mathbb{R})$ then the extended map must have a non-trivial kernel. However, if $f \in \operatorname{Ker}\Omega_f^{-1}$ then, for arbitrary $g \in \mathcal{H}^+(\mathbb{R})$ we have

$$0 = (\Omega_f^{-1}f, \Omega_f^* g)_{\mathcal{H}} = (((\Omega_f^*)^{-1})^* f, \Omega_f^* g)_{\mathcal{H}} = (f, g)_{\mathcal{H}^+(\mathbb{R})}$$

which is impossible by the arbitreriness of g unless $f = 0$. Now, since Ω_f^{-1} and $(\Omega_f^*)^{-1}$ are unbounded then $\mathbf{T}_F$ and $\hat{T}_F$ are necessarily unbounded. We note that the fact that Ω_f^{-1} and $(\Omega_f^*)^{-1}$ are unbounded implies that $\inf\sigma(\mathbf{T}_F^{-1}) = \inf\sigma(\hat{T}_F^{-1}) = 0$.

Let $R_{\mathbf{T}_F^{-1}}(z) = (z - \mathbf{T}_F^{-1})^{-1}$ and $R_{\hat{T}_F^{-1}}(z) = (z - \hat{T}_F^{-1})^{-1}$ be, respectively, the resolvents of $\mathbf{T}_F^{-1}$ and $\hat{T}_F^{-1}$. Using the identities $(z - \mathbf{T}_F^{-1})R_{\mathbf{T}_F^{-1}}(z) = 1$, $(z - \hat{T}_F^{-1})R_{\hat{T}_F^{-1}}(z) = 1$ and $\Omega_f(z - \mathbf{T}_F^{-1}) = (z - \hat{T}_F^{-1})\Omega_f$ and $\Omega_f^*(z - \hat{T}_F^{-1}) = (z - \mathbf{T}_F^{-1})\Omega_f^*$ it is easy to verify that

$$R_{\mathbf{T}_F^{-1}}(z) = z^{-1}\left(\Omega_f^* R_{\hat{T}_F^{-1}}(z)\Omega_f + 1\right) \tag{11}$$

$$R_{\hat{T}_F^{-1}}(z) = z^{-1}\left(\Omega_f R_{\mathbf{T}_F^{-1}}(z)\Omega_f^* + 1\right)\,. \tag{12}$$

Eqns. (11), (12) imply the equality of spectrum $\sigma(\mathbf{T}_F^{-1}) = \sigma(\hat{T}_F^{-1})$. Inverting $\mathbf{T}_F^{-1}$ and $\hat{T}_F^{-1}$ it follows that $\sigma(\mathbf{T}_F) = \sigma(\hat{T}_F)$. We note that, since $\mathbf{T}_F^{-1}$ and $\hat{T}_F^{-1}$ are contractive, one can also utilize the theory of contraction operators on Hilbert space, and especially the notion of characteristic functions, to prove the equality of the spectrums. This point of view is illuminating and is considered in the appendix where another proof of the equality of the spectrums is provided.

Let $\xi_{\mathcal{H}^+}$ be the spectral measure of $\hat{T}_F$ and, for each $E \in \mathfrak{B}^+$ let the operator $\xi_{\mathcal{H}}(E)$ be defined by the right hand side of Eq. (6). For any $u \in \operatorname{Ran}\Omega_f^*$ we have

$$\begin{aligned}\xi_{\mathcal{H}}^*(E)u = \Omega_f^*\xi_{\mathcal{H}^+}(E)(\Omega_f^{-1})^*u = \Omega_f^*\xi_{\mathcal{H}^+}(E)(\Omega_f^*)^{-1}\Omega_f^{-1}\Omega_f u = \Omega_f^*\xi_{\mathcal{H}^+}(E)\hat{T}_F\Omega_f u = \\ = \Omega_f^*\hat{T}_F\xi_{\mathcal{H}^+}(E)\Omega_f u = \Omega_f^{-1}\xi_{\mathcal{H}^+}(E)\Omega_f u = \xi_{\mathcal{H}}(E)u\,.\end{aligned}$$

Hence $\xi_{\mathcal{H}}(E)$ is symmetric on a dense set in $\mathcal{H}$ for any set $E \in \mathfrak{B}^+$. Furthermore, for any two sets $E_1, E_2 \in \mathfrak{B}^+$ it easy to see from the definition that we have

$$\xi_{\mathcal{H}}(E_1)\xi_{\mathcal{H}}(E_2) = \xi_{\mathcal{H}}(E_2 \cap I_1) \tag{13}$$

and for $E_1 \cap E_2 = \emptyset$

$$\xi_{\mathcal{H}}(E_1) + \xi_{\mathcal{H}}(E_2) = \xi_{\mathcal{H}}(E_1 \cup E_2)\,. \tag{14}$$

In particular, if we take in Eq. (13) $E_1 = E_2 = E$ we get

$$\xi_{\mathcal{H}}^2(E) = \xi_{\mathcal{H}}(E)\,.$$

Thus for each $E \in \mathfrak{B}^+$, $\xi_{\mathcal{H}}(E)$ is idempotent and symmetric on the dense set $\operatorname{Ran}\Omega_f^* \subset \mathcal{H}$ and can be extended uniquly to an orthogonal projection on $\mathcal{H}$. Eq. (13) and Eq. (14) imply that $\xi_{\mathcal{H}}$ is a spectral projection valued measure of a self-adjoint operator. Now $\operatorname{Dom}\mathbf{T}_F = \operatorname{Ran}\mathbf{T}_F^{-1} \subset \operatorname{Ran}\Omega_f^*$. Then for every $g \in \operatorname{Dom}\mathbf{T}_F$ we have

$$\int_{\sigma(\hat{T}_F)} \lambda\,\xi_{\mathcal{H}}(d\lambda)g = \Omega_f^{-1}\int_{\sigma(\hat{T}_F)} \lambda\,\xi(d\lambda)\Omega_f g = \Omega_f^{-1}\hat{T}_F\Omega_f g = \Omega_f^{-1}(\Omega_f^*)^{-1}g = \mathbf{T}_F g$$

hence $\xi_{\mathcal{H}}$ is the spectral measure of $\mathbf{T}_F$.

To verify Eq. (7) we note first that the right hand side of this equation is well defined on the dense set $\operatorname{Ran}\Omega_f$. This is so since we have $(\Omega_f^*)^{-1}\xi_{\mathcal{H}}(E)\Omega_f^*\Omega_f = (\Omega_f^*)^{-1}\xi_{\mathcal{H}}(E)\mathbf{T}_F^{-1} = (\Omega_f^*)^{-1}\mathbf{T}_F^{-1}\xi_{\mathcal{H}}(E) = \Omega_f\xi_{\mathcal{H}}(E)$. Plugging Eq. (6) into the right hand side of Eq. (7) we get on this dense set

$$\begin{aligned}&(\Omega_f^*)^{-1}\xi_{\mathcal{H}}(E)\Omega_f^* f = \\ &\qquad = (\Omega_f^*)^{-1}\Omega_f^{-1}\xi_{\mathcal{H}^+}(E)\Omega_f\Omega_f^* f = \hat{T}_F\xi_{\mathcal{H}^+}(E)\hat{T}_F^{-1}f = \xi_{\mathcal{H}^+}(E)f, \quad f \in \operatorname{Ran}\Omega_f\,.\end{aligned}$$

The proof of the statements in Proposition 1 concerning backward time observables can be obtained by following the same steps as for the forward time observables with the obvious replecments of Ω_f by Ω_b, $\mathbf{T}_F$ by $\mathbf{T}_B$ and $\hat{T}_F$ by $\hat{T}_B$. ■

Remark: It is useful to note that, with the help of Eq. (10) we can write $\mathbf{T}_F$, $\hat{T}_F$ and $\mathbf{T}_B$, $\hat{T}_B$ in a form exhibiting more clearly their positivity and symmetric nature, i.e.,

$$\mathbf{T}_F = \Omega_f^{-1}(\Omega_f^*)^{-1} = \Omega_f^{-1}(\Omega_f^{-1})^*, \quad \hat{T}_F = (\Omega_f^*)^{-1}\Omega_f^{-1} = (\Omega_f^{-1})^*\Omega_f^{-1}\,.$$

$$\mathbf{T}_B = \Omega_b^{-1}(\Omega_b^*)^{-1} = \Omega_b^{-1}(\Omega_b^{-1})^*, \quad \hat{T}_B = (\Omega_b^*)^{-1}\Omega_b^{-1} = (\Omega_b^{-1})^*\Omega_b^{-1}\,.$$

The origin of the terminology used for $\mathbf{T}_F$ and $\mathbf{T}_B$, i.e., our reference to them as time observables for forward and backward evolution respectively, follows from the next theorem

Theorem 2 *We have* $\inf\sigma(\mathbf{T}_F) = \inf\sigma(\mathbf{T}_B) = 1$. *Let* $\xi_{\mathcal{H}}$ *be the spectral measure of* $\mathbf{T}_F$ *and let* $a > 1$. *Then, for any* $g \in \mathcal{H}$ *satisfying* $\xi_{\mathcal{H}}([1,a))g = g$ *there exists a time* $\tau > 0$ *such that* $\xi_{\mathcal{H}}([a,\infty))\mathbf{U}(t)g \neq \{0\}$ *for all* $t > \tau$ *and, moreover,* $\lim_{t\to\infty}\|\xi_{\mathcal{H}}([1,a))\mathbf{U}(t)g\| = 0$. *Let* $\zeta_{\mathcal{H}}$ *be the spectral measure of* $\mathbf{T}_B$. *Then for any* $g \in \mathcal{H}$ *satisfying* $\zeta_{\mathcal{H}}([1,a))g = g$ *there exists a time* $\tau' < 0$ *such that* $\zeta_{\mathcal{H}}([a,\infty))\mathbf{U}(t)g \neq \{0\}$ *for all* $t < \tau'$ *and, moreover,* $\lim_{t\to-\infty}\|\zeta_{\mathcal{H}}([1,a))\mathbf{U}(t)g\| = 0$. □

Proof:

As above full details are given for the case of $\mathbf{T}_F$ with an indication of changes necessary for the case of T_B. In order to prove the first part of the theorem we need some more information on the structure of the operators Ω_f and Ω_b. Let $U : \mathcal{H} \mapsto L^2(\mathbb{R}^+)$ be the unitary mapping of $\mathcal{H}$ onto its spectral representation on the spectrum of $\mathbf{H}$ (the energy representation for $\mathbf{H}$). Let $P_{\mathbb{R}^+} : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ be the orthogonal projection in $L^2(\mathbb{R})$ on the subspace of functions supported on $\mathbb{R}^+$ and define the inclusion map $I : L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R})$ by

$$(If)(\sigma) = \begin{cases} f(\sigma), & \sigma \geq 0\,, \\ 0, & \sigma < 0\,. \end{cases} \tag{15}$$

Then the inverse $I^{-1} : P_{\mathbb{R}^+}L^2(\mathbb{R}) \mapsto L^2(\mathbb{R}^+)$ is well defined on $P_{\mathbb{R}^+}L^2(\mathbb{R})$. Let $\theta : \mathcal{H}^+(\mathbb{R}) \mapsto L^2(\mathbb{R}^+)$ be given by

$$\theta f = I^{-1}P_{\mathbb{R}^+}f, \qquad f \in \mathcal{H}^+(\mathbb{R})\,. \tag{16}$$

By a theorem of Van Winter [36], θ is a contractive quasi-affine transform mapping $\mathcal{H}^+(\mathbb{R})$ into $L^2(\mathbb{R}^+)$. The adjoint map $\theta^* : L^2(\mathbb{R}^+) \mapsto \mathcal{H}^+(\mathbb{R})$ is then a contractive quasi-affine map. An explicit expression for θ^* is given by [28]

$$\theta^* f = P_+ If, \qquad f \in L^2(\mathbb{R}^+)\,.$$

It is shown in [28] that the maps Ω_f, Ω_f^* are given by

$$\Omega_f = \theta^* U, \qquad \Omega_f^* = U^*\theta\,. \tag{17}$$

If in Eq. (16) instead of functions in $\mathcal{H}^+(\mathbb{R})$ we consider functions in $\mathcal{H}^-(\mathbb{R})$ we obtain instead of θ a different contractive quasi-affine map $\overline{\theta} : \mathcal{H}^-(\mathbb{R}) \mapsto L^2(\mathbb{R}^+)$

$$\overline{\theta} f = I^{-1}P_{\mathbb{R}^+}f, \qquad f \in \mathcal{H}^-(\mathbb{R})\,. \tag{18}$$

Then $\overline{\theta}^* : L^2(\mathbb{R}^+) \mapsto \mathcal{H}^-(\mathbb{R})$ is given by $\overline{\theta}^* f = P_- If$, $f \in L^2(\mathbb{R}^+)$ and, in a way similar to Eq. (17), one obtains

$$\Omega_b = \overline{\theta}^* U, \qquad \Omega_b^* = U^*\overline{\theta}\,.$$

In $\mathcal{H}^+(\mathbb{R})$ consider an element x_μ of the form $x_\mu(\lambda) = (\lambda - \mu)^{-1}$, $\operatorname{Im}\mu < 0$ and denote $\psi_\mu = \Omega_f^* x_\mu$. We have

$$\frac{(x_\mu, \hat{T}_F^{-1} x_\mu)_{\mathcal{H}^+(\mathbb{R})}}{\|x_\mu\|^2_{\mathcal{H}^+(\mathbb{R})}} = \frac{(x_\mu, \Omega_f \Omega_f^* x_\mu)_{\mathcal{H}^+(\mathbb{R})}}{\|x_\mu\|^2_{\mathcal{H}^+(\mathbb{R})}} = \frac{\|\psi_\mu\|^2_{\mathcal{H}}}{\|x_\mu\|^2_{\mathcal{H}^+(\mathbb{R})}} \,.$$

Therefore

$$\|\hat{T}_F^{-1}\| = \sup_{\|\varphi\|_{\mathcal{H}^+(\mathbb{R})}=1} (\varphi, \hat{T}_F^{-1}\varphi) \geq \frac{\|\psi_\mu\|^2_{\mathcal{H}}}{\|x_\mu\|^2_{\mathcal{H}^+(\mathbb{R})}} \,. \tag{19}$$

Furthermore, according to Eq. (16) and Eq. (17) we have

$$\|\psi_\mu\|^2_{\mathcal{H}} = \|U^*\theta x_\mu\|^2_{\mathcal{H}} = \|\theta x_\mu\|^2_{L^2(\mathbb{R}^+)} = \int_0^\infty d\lambda\, |\lambda - \mu|^{-2} \tag{20}$$

and

$$\|x_\mu\|^2_{\mathcal{H}^+(\mathbb{R})} = \int_{-\infty}^\infty d\lambda\, |\lambda - \mu|^{-2} = \frac{\pi}{|\operatorname{Im}\mu|} \,.$$

Now, changing variables in Eq. (20) we get that

$$\|\psi_\mu\|^2_{\mathcal{H}} = \int_{-\operatorname{Re}\mu}^\infty d\lambda\, |\lambda - i\operatorname{Im}\mu|^{-2}$$

and therefore for $\operatorname{Im}\mu$ constant we have $\lim_{\operatorname{Re}\mu\to-\infty} \|\psi_\mu\|^2 = 0$ and $\lim_{\operatorname{Re}\mu\to\infty} \|\psi_\mu\|^2 = \|x_\mu\|^2_{\mathcal{H}^+(\mathbb{R})}$. Hence, taking the limit $\operatorname{Re}\mu \to \infty$ in Eq. (19) with constant $\operatorname{Im}\mu$ we obtain that in fact $\|\hat{T}_F^{-1}\| = 1$ and, by the equality of the spectrum, $\|\mathbf{T}_F^{-1}\| = 1$. Thus, we get that $\sup\sigma(\mathbf{T}_F^{-1}) = \sup\sigma(\hat{T}_F^{-1}) = 1$ and therefore $\inf\sigma(\mathbf{T}_F) = \inf\sigma(\hat{T}_F) = 1$. The considerations for the case of the operators $\mathbf{T}_B$ and $\hat{T}_B$ are similar with the main difference being in the replacement of x_μ by a state $x'_\mu \in \mathcal{H}^-(\mathbb{R})$ of the form $x'_\mu(\lambda) = (\lambda - \overline{\mu})^{-1}$, $\operatorname{Im}\mu < 0$ and in the replacement of ψ_μ by the state $\psi'_\mu := \Omega_b^* x'_\mu$. One then arrives at the result that $\inf\sigma(\mathbf{T}_B) = \inf\sigma(\hat{T}_B) = 1$.

In order to prove the rest of the statements in the theorem we need the following lemma

Lemma 1 *The following intertwining relations hold for $t \geq 0$:*

$$\begin{aligned}
\Omega_f \mathbf{U}(t) &= T_u^+(t)\Omega_f & (21)\\
\mathbf{U}(t)\Omega_f^{-1}|_{Ran\,\Omega_f} &= \Omega_f^{-1} T_u^+(t)|_{Ran\,\Omega_f}, & (22)\\
\Omega_f^* T_u^{+*}(t) &= \mathbf{U}(-t)\Omega_f^*, & (23)\\
T_u^{+*}(t)(\Omega_f^*)^{-1}|_{Ran\,\Omega_f^*} &= (\Omega_f^*)^{-1}\mathbf{U}(-t)|_{Ran\,\Omega_f^*} \,. & (24)
\end{aligned}$$

□

Proof of Lemma 1:

Given the mapping $\Omega_f : \mathcal{H} \mapsto \mathcal{H}^+(\mathbb{R})$, consider the basic intertwining relation in Eq. (4). Using this equation we obtain

$$(\mathbf{U}(-t)\Omega_f^* g, f)_{\mathcal{H}} = (g, \Omega_f \mathbf{U}(t) f)_{\mathcal{H}^+(\mathbb{R})} = (g, T_u^+(t)\Omega_f f)_{H^+(\mathbb{R})} = (\Omega_f^* T_u^{+*}(t) g, f)_{\mathcal{H}},$$
$$\forall g \in \mathcal{H}^+(\mathbb{R}),\ \forall f \in \mathcal{H},\ t \geq 0$$

and so

$$\Omega_f^* T_u^{+*}(t) = \mathbf{U}(-t)\Omega_f^*, \qquad t \geq 0\,. \tag{25}$$

By the injective property of the quasi-affine mappings Ω_f and Ω_f^*, Eq. (22) and Eq. (24) are direct consequences of Eq. (4) and Eq. (25). ■

Denote

$$T_t(X) := (T_u^+(t))^* X T_u^+(t), \quad t \geq 0,\ X \in \mathcal{B}(\mathcal{H}^+(\mathbb{R}))$$

where $\mathcal{B}(\mathcal{H}^+(\mathbb{R}))$ is the space of bounded operators on $\mathcal{H}^+(\mathbb{R})$. Using Lemma 1 we obtain, for any $X \in \mathcal{B}(\mathcal{H}^+(\mathbb{R}))$ and $g \in \mathcal{H}$,

$$\Omega_f^* T_t(X)\Omega_f\, g = \Omega_f^*(T_u^+(t))^* X T_u^+(t)\Omega_f\, g = \mathbf{U}(-t)\Omega_f^* X \Omega_f \mathbf{U}(t)\, g\,.$$

Hence,

$$(\Omega_f g, T_t(X)\Omega_f g)_{\mathcal{H}^+(\mathbb{R})} = (\mathbf{U}(t)g, \Omega_f^* X \Omega_f \mathbf{U}(t) g)_{\mathcal{H}}\,. \tag{26}$$

Denote $g_+ := \Omega_f g$, $g_+(t) := T_u^+(t) g_+$ and $\tilde{g}_+(t) := g_+(t)\|g_+(t)\|_{\mathcal{H}^+}^{-1}$ and note that, for all $g \in \mathcal{H}$, $\|g_+(t_2)\|_{\mathcal{H}^+} \leq \|g_+(t_1)\|_{\mathcal{H}^+}$ for $t_2 \geq t_1$ and, for all $g \in \mathcal{H}$, $\lim_{t\to\infty}\|g_+(t)\|_{\mathcal{H}^+} = 0$. Then Eq. (26) can be written

$$\|g_+(t)\|_{\mathcal{H}^+}^2 (\tilde{g}_+(t), X\tilde{g}_+(t))_{\mathcal{H}^+} = (\mathbf{U}(t)g, \Omega_f^* X \Omega_f \mathbf{U}(t) g)_{\mathcal{H}}\,. \tag{27}$$

For the operator $X \in \mathcal{B}(\mathcal{H}^+(\mathbb{R}))$ in Eq. (27) consider the choice $X = \xi_{\mathcal{H}^+}([1,a))\hat{T}_F$ (recall that $\xi_{\mathcal{H}^+}$ is the spectral measure of $\hat{T}_F$). For this choice of X Eq. (27) reads

$$\begin{aligned}
&\|g_+(t)\|_{\mathcal{H}^+}^2 (\tilde{g}_+(t), \xi_{\mathcal{H}^+}([1,a))\hat{T}_F \xi_{\mathcal{H}^+}([1,a))\tilde{g}_+(t))_{\mathcal{H}^+} = \\
&\qquad = (\mathbf{U}(t)g, \Omega_f^* \hat{T}_F \xi_{\mathcal{H}^+}([1,a))\Omega_f \mathbf{U}(t) g)_{\mathcal{H}} = \\
&\qquad = (\mathbf{U}(t)g, \Omega_f^{-1} \xi_{\mathcal{H}^+}([1,a))\Omega_f \mathbf{U}(t) g)_{\mathcal{H}} = (\mathbf{U}(t)g, \xi_{\mathcal{H}}([1,a))\mathbf{U}(t)g)_{\mathcal{H}}\,.
\end{aligned} \tag{28}$$

Choose any $g \in \mathcal{H}$ such that $g = \xi_{\mathcal{H}}([1,a))g$. If we also have $\xi_{\mathcal{H}}([1,a))\mathbf{U}(t)g = \mathbf{U}(t)g$ for all $t \geq 0$ then the right hand side of Eq. (28) is equal to $\|g\|_{\mathcal{H}}^2$ for all $t \geq 0$. However, $\|g_+(t)\|_{\mathcal{H}^+}$ is non-increasing, $\lim_{t\to\infty}\|g_+(t)\|_{\mathcal{H}^+} = 0$ and $\|\tilde{g}_+(t)\|_{\mathcal{H}^+} = 1$, hence there must exist a time $\tau > 0$ such that

$$\|g_+(t)\|_{\mathcal{H}^+}^2 \left(\sup_{\substack{t\geq 0\\ g\in\mathcal{H}}} \left| (\tilde{g}_+(t), \xi_{\mathcal{H}^+}([1,a))\hat{T}_F \xi_{\mathcal{H}^+}([1,a))\tilde{g}_+(t))_{\mathcal{H}^+} \right| \right) < \|g\|_{\mathcal{H}}^2, \quad t > \tau\,. \tag{29}$$

The contradiction thus obtained implies that $\xi_{\mathcal{H}}([a,\infty))\mathbf{U}(t)g \neq \{0\}$ for all $t > \tau$. Furthermore, since the left hand side of Eq. (28) vanishes in the limit as t goes to infinity we must have $\lim_{t\to\infty}\|\xi_{\mathcal{H}}([1,a))\mathbf{U}(t)g\|_{\mathcal{H}} = 0$.

The proof of the last statement in Theorem 2 concerning the operator $\mathbf{T}_B$ is similar to the proof above for $\mathbf{T}_F$. ■

Theorem 2 motivates our point of view of $\mathbf{T}_F$ as being a time observable for the quantum evolution in the forward direction since, *for $g \in \mathcal{H}$, $\mathbf{U}(t)g$ must "go up" on the spectrum of $\mathbf{T}_F$ as time increases.* We note that the rate of flow of the evolved state up on the spectrum of $\mathbf{T}_F$ depends on the choice of the state g.

2.2 The origin of the time observables

It is clear from the proof of Theorem 2, and in particular from Eq. (28) and the definition of $g_+(t)$, that the existence of the time observables and the fundamental time asymmetry inherent in the definition of distinct backward and forward time observables is a direct consequence of the the fundamental intertwining relations in Eqns. (4), (5). In fact, we can do better and show that further analysis of this basic equation provides a more clear understanding of the origin of the time observables. We take up this task in this subsection (the discussion here partly follows Section (IV) of reference [29]).

Let $\mathcal{S}$ be the *Schwartz class* of rapidly decreasing functions in $C^\infty(\mathbb{R})$ and let $\mathcal{S}'$ be the space of *tempered distributions* on $\mathcal{S}$. For any fixed $p \in (0, \infty)$ let $\mathcal{H}^p(\mathbb{C}\backslash\mathbb{R})$ be the space of analytic functions on $\mathbb{C}\backslash\mathbb{R}$ for which

$$\|f\| = \sup_{y \neq 0} \left(\int_{\mathbb{R}} |f(x+iy)|^p \right)^{1/p} < \infty\,.$$

It can be shown [8] that every function $F \in \mathcal{H}^p(\mathbb{C}\backslash\mathbb{R})$ is associated with a unique tempered distribution $\ell_F \in \mathcal{S}'$ defined by

$$\ell_F(\psi) = \lim_{y \to 0^+} \int_{\mathbb{R}} \{F(x+iy) - F(x-iy)\}\psi(x)dx, \qquad \psi \in \mathcal{S}\,. \tag{30}$$

We denote the set of all such distributions by $H^p(\mathbb{R})$. Conversely, to any distribution $\ell \in H^p(\mathbb{R})$ with $p \in (0, \infty)$ we can associate a unique function $F_\ell \in \mathcal{H}^p(\mathbb{C}\backslash\mathbb{R})$ such that

$$\ell_{F_\ell}(\psi) = \lim_{y \to 0^+} \int_{\mathbb{R}} \{F_\ell(x+iy) - F_\ell(x-iy)\}\psi(x)dx = \ell(\psi), \quad \psi \in \mathcal{S}\,.$$

The function $F_\ell \in \mathcal{H}^p(\mathbb{C}\backslash\mathbb{R})$ is then given by [8]

$$F_\ell(z) = \frac{1}{2\pi i}\ell((\cdot - z)^{-1}), \qquad z \in \mathbb{C}\backslash\mathbb{R}\,. \tag{31}$$

Now, for $p \in (1, \infty)$ we have the further identification of the space of distributions $H^p(\mathbb{R})$ with the function space $L^p(\mathbb{R})$ in the sense that any function $f \in L^p(\mathbb{R})$ defines a tempered distribution on $\mathcal{S}$ via

$$\ell_f(\psi) = \int_{\mathbb{R}} f(x)\psi(x)\,dx, \qquad \psi \in \mathcal{S} \tag{32}$$

and Eq. (31) associates with f a unique analytic function $F_{\ell_f} \in \mathcal{H}^p(\mathbb{C}\backslash\mathbb{R})$, i.e.,

$$F_{\ell_f}(z) = \frac{1}{2\pi 1}\ell_f((\cdot - z)^{-1}) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(x)}{x-z}\,dx\,. \tag{33}$$

Using Eq. (30) we can then recover the distribution ℓ_f from the function F_{ℓ_f}, i.e., we have $\ell_{F_{\ell_f}} = \ell_f$. For our purpose we need also the following proposition [8]:

Proposition 2 *A distribution $\ell \in H^p(\mathbb{R})$ has support which omits an open interval $\Delta \in \mathbb{R}$ iff the corresponding function $F_\ell \in \mathcal{H}^p(\mathbb{C}\backslash\mathbb{R})$ given by Eq. (31) has an anlytic continuation across the interval Δ.*

We now restrict the discussion above to the case $p = 2$ and consider the embedding $I : L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R})$ in Eq. (15). For any $f \in L^2(\mathbb{R}^+)$ we have $If \in L^2(\mathbb{R})$. Using the identification of $L^2(\mathbb{R})$ with the space $H^2(\mathbb{R})$ as above we associate with If, through Eq. (33), the function $F_{\ell_{If}} \in \mathcal{H}^p(\mathbb{C}\backslash\mathbb{R})$. Clearly the distribution defined by If omits the interval $\mathbb{R}^-$ and therefore, by Proposition 2, $F_{\ell_{If}}$ is analytic across the neagtive real axis i.e, $F_{\ell_{If}} \in \mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+) \subset \mathcal{H}^2(\mathbb{C}\backslash\mathbb{R})$ where $\mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+)$ denotes the subspace of $\mathcal{H}^2(\mathbb{C}\backslash\mathbb{R})$ containing functions analytic across $\mathbb{R}^-$. We shall use the notation $F_f \equiv F_{\ell_{If}}$.

Note that Eq. (30) and the uniqueness of the functional ℓ_f in Eq. (32) allows us to associate with each function $F \in \mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+)$ the corresponding function $f \in L^2(\mathbb{R}^+)$. Thus we have the following lemma

Lemma 2 *There exists a bijective map $A' : L^2(\mathbb{R}^+) \mapsto \mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+)$ such that $A'f = F_f$ with $f \in L^2(\mathbb{R}^+)$, $F_f \in \mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+)$, F_f given by*

$$F_f(z) = \frac{1}{2\pi i}\int_{\mathbb{R}^+} \frac{f(x)}{x-z}\,dx, \qquad f \in L^2(\mathbb{R}^+)\,,$$

and, given F_f we have

$$f = A'^{-1}F_f = I^{-1}[F_f^+ - F_f^-] \tag{34}$$

where

$$F_f^+(\sigma) = \lim_{\epsilon\to 0^+} F_f(\sigma + i\epsilon), \qquad F_f^-(\sigma) = \lim_{\epsilon\to 0^+} F_f(\sigma - i\epsilon) \qquad \sigma \in \mathbb{R}\,. \tag{35}$$

and we note that the boundary value functions F_f^+ and F_f^- exist a.e. since the restriction of F_f to $\mathbb{C}^+$ is an element of $\mathcal{H}^2(\mathbb{C}^+)$ and the restriction of F_f to $\mathbb{C}^-$ is an element of $\mathcal{H}^2(\mathbb{C}^-)$.
□

Proof:

In view of the discussion above we have only to find an explicit form for the transformation A. This is obtained through the use of Eq. (33) with the result

$$F_f(z) \equiv F_{\ell_{If}}(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{If(x)}{x-z}\,dx = \frac{1}{2\pi i}\int_{\mathbb{R}^+} \frac{f(x)}{x-z}\,dx, \qquad f \in L^2(\mathbb{R}^+)\,.$$

In addition Eq. (34) is a direct result of Eq. (30) and Eq. (32). ■

Consider the unitary map $U : \mathcal{H} \mapsto L^2(\mathbb{R}^+)$ mapping $\mathcal{H}$ onto its energy representation on the spectrum of $\mathbf{H}$. Combining the mappings U and A' we get a bijective map $A : \mathcal{H} \mapsto \mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+)$ with $A =: A'U$. For an element $\psi \in \mathcal{H}$ denote $\psi_A = A\psi = A'U\psi$. Choosing an element $\psi \in \mathcal{H}$ as an initial state and letting it evolve under the Schrödinger evolution $\mathbf{U}(t)$ we get an induced evolution in $\mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+)$

$$\psi_A(t) = A\psi_t = A'U\psi_t = A'U\mathbf{U}(t)\psi\,.$$

We would like to characterize this induced evolution. Denote by $\psi_A^+(t)$ the restriction of $\psi_A(t)$ to $\mathbb{C}^+$ and note that $\psi_A^+(t) \in \mathcal{H}^2(\mathbb{C}^+)$. Similarly, if $\psi_A^-(t)$ denotes the restriction of $\psi_A(t)$ to $\mathbb{C}^-$ then $\psi_A^-(t) \in \mathcal{H}^2(\mathbb{C}^-)$. For each time t we have

$$\|\psi_A^+(t)\|_{\mathcal{H}^2(\mathbb{C}^+)} < \infty, \quad \|\psi_A^-(t)\|_{\mathcal{H}^2(\mathbb{C}^-)} < \infty\,.$$

Recall that for an element $f \in L^2(\mathbb{R}^+)$ the boundary value functions F_f^+ and F_f^- (see Eq. (35) above) belong, repectively, to $\mathcal{H}^+(\mathbb{R})$ and $\mathcal{H}^-(\mathbb{R})$. Considering the mappings $\theta^* : L^2(\mathbb{R}^+) \mapsto \mathcal{H}^+(\mathbb{R})$ and $\overline{\theta}^* : L^2(\mathbb{R}^+) \mapsto \mathcal{H}^-(\mathbb{R})$ for any function $f \in L^2(\mathbb{R}^+)$ we have

$$If = P_+ If + P_- If = \theta^* f + \overline{\theta}^* f\,. \tag{36}$$

Since $L^2(\mathbb{R}) = \mathcal{H}^+(\mathbb{R}) \oplus \mathcal{H}^-(\mathbb{R})$ the sum in Eq. (36) is unique. However, from Eq. (35) we obtain

$$If = IA'^{-1} F_f = F_f^+ - F_f^-\,.$$

hence $F_f^+ = \theta^* f$ and $F_f^- = -\overline{\theta}^* f$. Denote by $F^+_{\psi_A(t)}$ the boundary value of $\psi_A^+(t)$ on $\mathbb{R}$ and by $F^-_{\psi_A(t)}$ the boundary value of $\psi_A^-(t)$ on $\mathbb{R}$. Then $F^+_{\psi_A(t)} \in \mathcal{H}^+(\mathbb{R})$ and $F^-_{\psi_A(t)} \in \mathcal{H}^-(\mathbb{R})$ and we have, for any $t \in \mathbb{R}$

$$F^+_{\psi_A(t)} = \theta^* U \psi_t = \Omega_f \mathbf{U}(t)\psi$$

and

$$F^-_{\psi_A(t)} = -\overline{\theta}^* U \psi_t = -\Omega_b \mathbf{U}(t)\psi\,.$$

Using Eq. (4) and the isomorphism of $\mathcal{H}^2(\mathbb{C}^+)$ and $\mathcal{H}^+(\mathbb{R})$ we get

$$\|\psi_A^+(t)\|_{\mathcal{H}^2(\mathbb{C}^+)} = \|F^+_{\psi_A(t)}\|_{\mathcal{H}^+(\mathbb{R})} = \|\Omega_f \mathbf{U}(t)\psi\|_{\mathcal{H}^+(\mathbb{R})} = \|T_u^+(t)\Omega_f\psi\|_{\mathcal{H}^+(\mathbb{R})}, \quad t \geq 0\,. \tag{37}$$

We conclude that, for $t \geq 0$, $\|\psi_A^+(t)\|_{\mathcal{H}^2(\mathbb{C}^+)}$ is monotonically decreasing and, furthermore, $\lim_{t\to\infty} \|\psi_A^+(t)\|_{\mathcal{H}^2(\mathbb{C}^+)} = 0$. Because of the fact that $\|\psi_A^+(t)\|_{\mathcal{H}^2(\mathbb{C}^+)} + \|\psi_A^-(t)\|_{\mathcal{H}^2(\mathbb{C}^-)} = \|\psi\|_{\mathcal{H}}$ we obtain that $\|\psi_A^-(t)\|_{\mathcal{H}^2(\mathbb{C}^-)}$ is monotonically increasing for $t \geq 0$ and we have $\lim_{t\to\infty} \|\psi_A^-(t)\|_{\mathcal{H}^2(\mathbb{C}^-)} = \|\psi\|_{\mathcal{H}}$. Using Eq. (5) and the isomorphism of $\mathcal{H}^2(\mathbb{C}^-)$ and $\mathcal{H}^-(\mathbb{R})$ we obtain

$$\|\psi_A^-(t)\|_{\mathcal{H}^2(\mathbb{C}^-)} = \|F^-_{\psi_A(t)}\|_{\mathcal{H}^-(\mathbb{R})} = \|\Omega_b \mathbf{U}(t)\psi\|_{\mathcal{H}^-(\mathbb{R})} = \|T_u^-(t)\Omega_b\psi\|_{\mathcal{H}^-(\mathbb{R})}, \quad t \leq 0\,. \tag{38}$$

and we conclude that, for $t \leq 0$, $\|\psi_A^-(t)\|_{\mathcal{H}^2(\mathbb{C}^-)}$ is monotonically decreasing and we have $\lim_{t\to-\infty} \|\psi_A^-(t)\|_{\mathcal{H}^2(\mathbb{C}^-)} = 0$. Moreover, $\|\psi_A^+(t)\|_{\mathcal{H}^2(\mathbb{C}^+)}$ is monotonically increasing for $t \leq 0$ and $\lim_{t\to-\infty} \|\psi_A^+(t)\|_{\mathcal{H}^2(\mathbb{C}^+)} = \|\psi\|_{\mathcal{H}}$. Summarising, we have proved the following

Proposition 3 *There exists a bijective map $A : \mathcal{H} \mapsto \mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+)$ with $A = A'U$, $U : \mathcal{H} \mapsto L^2(\mathbb{R}^+)$ is the mapping of $\mathcal{H}$ onto its energy representation on the spectrum of $\mathbf{H}$ and $A' : L^2(\mathbb{R}^+) \mapsto \mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+)$ defined in Lemma 2. For $\psi \in \mathcal{H}$ denote $\psi_A = A\psi$ and $\psi_A(t) = A\mathbf{U}(t)\psi$. Denote $\psi_A^+(t)$ the restriction of $\psi_A(t)$ to $\mathbb{C}^+$ and $\psi_A^-(t)$ the restriction of $\psi_A(t)$ to $\mathbb{C}^-$. Then we have*

$$\|\psi_A^+(t_1)\|_{\mathcal{H}^2(\mathbb{C}^+)} \geq \|\psi_A^+(t_2)\|_{\mathcal{H}^2(\mathbb{C}^+)}, \quad 0 \leq t_1 < t_2\,,$$

$$\|\psi_A^-(t_1)\|_{\mathcal{H}^2(\mathbb{C}^-)} \leq \|\psi_A^-(t_2)\|_{\mathcal{H}^2(\mathbb{C}^-)}, \quad 0 \leq t_1 < t_2$$

and

$$\lim_{t\to\infty} \|\psi_A^+(t)\|_{\mathcal{H}^2(\mathbb{C}^+)} = 0, \quad \lim_{t\to\infty} \|\psi_A^-(t)\|_{\mathcal{H}^2(\mathbb{C}^-)} = \|\psi\|_{\mathcal{H}}\,.$$

In addition

$$\|\psi_A^+(t_1)\|_{\mathcal{H}^2(\mathbb{C}^+)} \leq \|\psi_A^+(t_2)\|_{\mathcal{H}^2(\mathbb{C}^+)}, \quad t_2 < t_1 \leq 0\,,$$

$$\|\psi_A^-(t_1)\|_{\mathcal{H}^2(\mathbb{C}^-)} \geq \|\psi_A^-(t_2)\|_{\mathcal{H}^2(\mathbb{C}^-)}, \quad t_2 < t_1 \leq 0$$

and

$$\lim_{t\to-\infty}\|\psi_A^-(t)\|_{\mathcal{H}^2(\mathbb{C}^-)}=0, \quad \lim_{t\to-\infty}\|\psi_A^+(t)\|_{\mathcal{H}^2(\mathbb{C}^+)}=\|\psi\|_{\mathcal{H}}\,.$$

□

It is the flow of norm from the upper half-plane Hardy space to the lower half-plane Hardy space induced by the Schrödinger evolution for positive times that gives rise to the time observable for forward evolution. In fact, as is evident from the proof of Proposition 3, this flow of norm provides the basic intertwining relation Eq. (4) which in turn stands at the heart of the proof of Theorem 2. In a similar manner, for negative times the Schrödinger evolution induces the flow of norm from the lower half-plane Hardy space to the upper half-plane Hardy space which gives rise to the time observable for backward evolution.

3 Time observables for quantum stochastic processes

3.1 Mapping of creation, annihilation and conservation processes

As mentioned in Section 1 there exists an inherent time asymmetry built into the semigroup decomposition formalism in the form of two distinct semigroup evolutions appearing in Eq. (4) and Eq. (5) and corresponding respectively to future directed evolution and to past directed evolution and in the existence of distinct forward and backward time observables $\mathbf{T}_F$ and $\mathbf{T}_B$. Considering for the moment forward time evolution (the treatment of backward evolution parallels the developments below) one may ask, in light of the discussion in Section 2, whether the use of $\mathbf{T}_F$ can be extended in such a way that it may serve a universal role as a forward time observable for more general quantum processes. Following this line of thought we will consider in this section the role of $\mathbf{T}_F$ as a time observable for quantum stochastic processes. We shall work in the setting of quantum stochastic differential equations defined in the framework of the Hudson-Parthasarathy (HP) quantum stochastic calculus. The terminology and notation below closely follows that of [25].

A simple answer to the question whether $\mathbf{T}_F$ can be applied as a time observable for quantum stochastic processes is: yes. This stems from the fact that on the abstract level a general $\mathbb{R}^+$-valued observable, i.e., a self-adjoint operator with spectral projection valued measure ξ defined on the Borel σ-algebra $\mathfrak{B}^+$, can be used as a time observable with respect to which one may define ξ-martingales and basic regular adapted processes which are then utilized for the definition of stochastic integration and the construction of quantum stochastic differential equations [17]. This abstract requirement is, however, not informative in the sense that it gives no characterization of the nature of the self-adjoint operator playing the role of a time observable. Therefore, a more concrete question is whether one may find the analogue of the fundamental intertwining relation in Eq. (4) (and Eq. (5) for the backward case). In other words one may ask whether it is possible to find a map associated with Ω_f that intertwines a (quantum) stochastic process, defined with respect to the physical Hilbert space $\mathcal{H}$ and the time observable $\mathbf{T}_F$, with a (quantum) stochastic process defined with respect to the Hardy space $\mathcal{H}^+(\mathbb{R})$ and the observable $\hat{T}_F$. We address this question in the present section.

As in Section 2 above let $\xi_{\mathcal{H}}:\mathfrak{B}^+\mapsto\mathcal{P}(\mathcal{H})$ be the spectral measure of $\mathbf{T}_F$ and $\xi_{\mathcal{H}^+}:\mathfrak{B}^+\mapsto\mathcal{P}(\mathcal{H}^+(\mathbb{R}))$ be the spectral measure of $\hat{T}_F$. The first step in the construction of fundamental adapted processes which respect to which stochastic integration can be defined is the definition of $\xi_{\mathcal{H}}$-martingales and of $\xi_{\mathcal{H}^+}$-martingales. We first recall the definition of

ξ-martingales. Let $\mathcal{K}$ be a complex separable Hilbert space and let $\xi : \mathfrak{B}^+ \mapsto \mathcal{P}(\mathcal{K})$ be a fixed $\mathbb{R}^+$-valued observable. For $0 \le s < t$ we define

$$\mathcal{K}_{t]} := \xi([0,t])\mathcal{K}, \quad \mathcal{K}_{[s,t]} := \xi([s,t])\mathcal{K}, \quad \mathcal{K}_{[t} := \xi([t,\infty))\mathcal{K}.$$

Then a ξ-martingale on $\mathcal{K}$ is defined as follows

Definition 4 *Let ξ be an $\mathbb{R}^+$-valued observable on $\mathcal{K}$. Let $m : \mathbb{R}^+ \mapsto \mathcal{K}$ be a map and for $t \in \mathbb{R}^+$ denote $m(t) \equiv m_t$. If the map m satisfies:*

1. $m_t \in \mathcal{K}_{t]}$, $\forall t \ge 0$,
2. $\xi([0,s])m_t = m_s$, $s < t$,

then m is called a ξ-martingale. □

In the case of the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^+(\mathbb{R})$ and the time observables $\mathbf{T}_F$ and $\hat{T}_F$ we shall use the notation $\mathcal{H}_{t]} = \xi_{\mathcal{H}}([1,t+1])\mathcal{H}$, $\mathcal{H}_{[s,t]} = \xi_{\mathcal{H}}([s+1,t+1])\mathcal{H}$, $\mathcal{H}_{[t} = \xi_{\mathcal{H}}([t+1,\infty))\mathcal{H}$ and $\mathcal{H}^+_{t]} = \xi_{\mathcal{H}^+}([1,t+1])\mathcal{H}^+(\mathbb{R})$, $\mathcal{H}^+_{[s,t]} = \xi_{\mathcal{H}^+}([s+1,t+1])\mathcal{H}^+(\mathbb{R})$, $\mathcal{H}^+_{[t} = \xi_{\mathcal{H}^+}([t+1,\infty))\mathcal{H}^+(\mathbb{R})$. Then, a $\xi_{\mathcal{H}}$-martingale is defined as in Definition 4 with conditions (.1)-(.2) adjusted in the form

$(1.)_{\mathcal{H}}$ $m_t \in \mathcal{H}_{t]}$, $\forall t \ge 0$

$(2.)_{\mathcal{H}}$ $\xi_{\mathcal{H}}([1,s+1])m_t = m_s$, $s < t$,

and a $\xi_{\mathcal{H}^+}$-martingale is defined as in Definition 4 with conditions (.1)-(.2) adjusted in the form

$(1.)_{\mathcal{H}}$ $m_t \in \mathcal{H}^+_{t]}$, $\forall t \ge 0$

$(2.)_{\mathcal{H}}$ $\xi_{\mathcal{H}^+}([1,s+1])m_t = m_s$, $s < t$,

Having defined $\xi_{\mathcal{H}}$-martingales and $\xi_{\mathcal{H}^+}$-martingales we have the following lemma concerning the mappings of martingales

Lemma 3 *Let m be a $\xi_{\mathcal{H}}$-martingale. Then the map $\hat{m} : \mathbb{R}^+ \mapsto \mathcal{H}^+(\mathbb{R})$ defined by $\hat{m}_t \equiv \hat{m}(t) := \Omega_f\, m_t$, $t \ge 0$ is a $\xi_{\mathcal{H}^+}$-martingale.* □

Proof of lemma 3:

This lemma is a result of Eq. (6) and Eq. (7) in Proposition 1 and the following simple calculations

$$\Omega_f\, \mathcal{H}_{t]} = \Omega_f \xi_{\mathcal{H}}([1,t+1])\mathcal{H} = \Omega_f \xi_{\mathcal{H}}([1,t+1])\Omega_f^{-1}\Omega_f\mathcal{H} = \xi_{\mathcal{H}^+}([1,t+1])\Omega_f\mathcal{H} \subset \mathcal{H}^+_{t]}\,. \quad (39)$$

and

$$\begin{aligned}\xi_{\mathcal{H}^+}([1,s+1])\hat{m}_t = (\Omega_f^*)^{-1}\xi_{\mathcal{H}}([1,s+1])\Omega_f^*\Omega_f m_t = (\Omega_f^*)^{-1}\xi_{\mathcal{H}}([1,s+1])\mathbf{T}_F^{-1} m_t = \\ = (\Omega_f^*)^{-1}\mathbf{T}_F^{-1}\xi_{\mathcal{H}}([1,s+1])m_t = \Omega_f m_s = \hat{m}_s\,. \quad (40)\end{aligned}$$

Eq. (39) shows that if $m_t \in \mathcal{H}_{t]}$ then $\hat{m}_t \in \mathcal{H}^+_{t]}$ in agreement with condition (1) in Definition 4. Eq. (40) corresponds to condition (2) in Definition 4. Hence $\hat{m}$ is a $\xi_{\mathcal{H}^+}$-martingale. □.

We now use the mapping of martingales given in Lemma 3 to map elementary adapted stochastic processes. We need first the following lemma

Lemma 4 *Let* $\mathbf{K} \in \mathcal{B}(\mathcal{H})$ *satisfy* $[\mathbf{K}, \xi_{\mathcal{H}}(E)] = 0$, $\forall E \in \mathfrak{B}^+$. *Define*

$$\hat{K} := \overline{(\Omega_f^*)^{-1}\mathbf{K}\Omega_f^*}\,. \tag{41}$$

Then $\hat{K} \in \mathcal{B}(\mathcal{H}^+(\mathbb{R}))$ *and* $[\hat{K}, \xi_{\mathcal{H}^+}(E)] = 0$, $\forall E \in \mathfrak{B}^+$. $\square$

Proof of Lemma 4:

Denote $|\Omega_f| = (\Omega_f^*\Omega_f)^{1/2} = (\mathbf{T}_F^{-1})^{1/2}$. Consider the map $|\Omega_f|^{-1}\Omega_f^*$ and note that

$$(|\Omega_f|^{-1}\Omega_f^*)^* = \Omega_f|\Omega_f|^{-1} = (\Omega_f^*)^{-1}\Omega_f^*\Omega_f|\Omega_f|^{-1} = (\Omega_f^*)^{-1}|\Omega_f|\,. \tag{42}$$

Note further that the map $|\Omega_f|^{-1}\Omega_f^*$ is well defined on the dense set $\operatorname{Ran}\Omega_f \subset \mathcal{H}^+(\mathbb{R})$. Using Eq. (42) it is clear that this map be extended to a unitary map X from $\mathcal{H}^+(\mathbb{R})$ to $\mathcal{H}$ (see [32]). The adjoint X^* is then an extension of the right hand side of Eq. (42). Using the unitary extension X we define an operator $K' := X^*\mathbf{K}X$. Obviously we have $K' \in \mathcal{B}(\mathcal{H}^+(\mathbb{R}))$. By assumption $\mathbf{K}$ commutes with the spectral measure $\xi_{\mathcal{H}}$ and hence with $|\Omega_f|$. Thus, for any $f \in \operatorname{Ran}\Omega_f$ we obtain with the help of Eq. (42)

$$K'f = X^*\mathbf{K}Xf = \Omega_f|\Omega_f|^{-1}\mathbf{K}|\Omega_f|^{-1}\Omega_f^*f = (\Omega_f^*)^{-1}\mathbf{K}\Omega_f^*f\,.$$

Hence we get that $\hat{K} = \overline{(\Omega_f^*)^{-1}\mathbf{K}\Omega_f^*} = K' \in \mathcal{B}(\mathcal{H}^+(\mathbb{R}))$. Moreover, using Eq.(7) we have

$$\begin{aligned}\xi_{\mathcal{H}^+}(E)(\Omega_f^*)^{-1}\mathbf{K}\Omega_f^*f = (\Omega_f^*)^{-1}\xi_{\mathcal{H}}(E)\Omega_f^*(\Omega_f^*)^{-1}\mathbf{K}\Omega_f^* = (\Omega_f^*)^{-1}\xi_{\mathcal{H}}(E)\mathbf{K}\Omega_f^* = \\ = (\Omega_f^*)^{-1}\mathbf{K}\xi_{\mathcal{H}}(E)\Omega_f^* = (\Omega_f^*)^{-1}\mathbf{K}\Omega_f^*\xi_{\mathcal{H}^+}(E)\,.\end{aligned} \tag{43}$$

The commutation relation $[\hat{K}, \xi_{\mathcal{H}^+}(E)] = 0$ is then obtained by taking the closure of the operator $(\Omega_f^*)^{-1}\mathbf{K}\Omega_f^*$ in Eq. (43). $\blacksquare$

Let $\mathcal{H}_0$ be a complex separable Hilbert space and let $\Gamma_s(\mathcal{H})$ be the symmetric (Bosonic) Fock space over $\mathcal{H}$ and $\Gamma_s(\mathcal{H}^+(\mathbb{R}))$ be the symmetric Fock space over $\mathcal{H}^+(\mathbb{R})$ [7, 25]. Denote

$$\tilde{\mathcal{H}} = \mathcal{H}_0 \otimes \Gamma_s(\mathcal{H}), \qquad \tilde{\mathcal{H}}^+ = \mathcal{H}_0 \otimes \Gamma_s(\mathcal{H}^+(\mathbb{R}))\,. \tag{44}$$

Below we consider in $\tilde{\mathcal{H}}$ *regular adapted processes* with respect to the triplet $(\xi_{\mathcal{H}}, \mathcal{H}_0, \mathcal{H})$ and in $\tilde{\mathcal{H}}^+$ regular adapted processes with respect to the triplet $(\xi_{\mathcal{H}^+}, \mathcal{H}_0, \mathcal{H}^+(\mathbb{R}))$ [17, 25]. Denote by $a(u)$ the *annihilation operator* and by $a^\dagger(u)$ the *creation operator* in $\Gamma_s(\mathcal{H})$ associated with $u \in \mathcal{H}$ and, for $\mathbf{K} \in \mathcal{B}(\mathcal{H})$, denote by $\lambda(\mathbf{K})$ the *conservation operator* associated with $\mathbf{K}$ [7, 25]. The annihilation, creation and conservation operators in $\Gamma_s(\mathcal{H}^+(\mathbb{R}))$ are denoted respectively by $\hat{a}(u)$, $\hat{a}^\dagger(u)$ and $\hat{\lambda}(\hat{K})$ where $u \in \mathcal{H}^+(\mathbb{R})$ and $\hat{K} \in \mathcal{B}(\mathcal{H}^+(\mathbb{R}))$. For u in $\mathcal{H}$ we denote by $e(u)$ the *exponential vector* in $\Gamma_s(\mathcal{H})$ associated with u and by $\mathcal{E}(\mathcal{H})$ the linear manifold generated by $\{e(u) \mid u \in \mathcal{H}\}$. The analogous objects in $\Gamma_s(\mathcal{H}^+(\mathbb{R}))$ are denoted $\hat{e}(u)$ and $\mathcal{E}(\mathcal{H}^+(\mathbb{R}))$. Note that $\{e(u) \mid u \in \mathcal{H}\}$ is *total* in $\Gamma_s(\mathcal{H})$ and $\{e(u) \mid u \in \mathcal{H}^+(\mathbb{R})\}$ is total in $\Gamma_s(\mathcal{H}^+(\mathbb{R}))$.

Let $\mathcal{W}$ be the algebra of operators on $\Gamma_s(\mathcal{H})$ generated by $\{a(u), a^\dagger(u), \lambda(\mathbf{K}), \mathbf{1}_{\Gamma_s(\mathcal{H})} \mid u \in \mathcal{H}, \mathbf{K} \in \mathcal{B}(\mathcal{H}), [\mathbf{K}, \mathbf{T}_F] = 0\}$. Let $\hat{\mathcal{W}}$ be the corrsponding algebra on $\Gamma_s(\mathcal{H}^+(\mathbb{R}))$ generated by $\{\hat{a}(u), \hat{a}^\dagger(u), \hat{\lambda}(\hat{K}), \mathbf{1}_{\Gamma_s(\mathcal{H}^+)} \mid u \in \mathcal{H}^+(\mathbb{R}), \hat{K} \in \mathcal{B}(\mathcal{H}^+(\mathbb{R})), [\hat{K}, \hat{T}_F] = 0\}$. Then the map $\Omega_f : \mathcal{H} \mapsto \mathcal{H}^+(\mathbb{R})$ induces a transformation $\Gamma(\Omega_f) : \mathcal{W} \mapsto \hat{\mathcal{W}}$ via the definition

Definition 5 ($\Gamma(\Omega_f)$, mapping of the algebra) *Let* $\mathbf{K}$ *be an operator in* $\mathcal{B}(\mathcal{H})$. *Define the operator* $\hat{K}$ *according to Eq. (41) i.e.,* $\hat{K} = \overline{(\Omega_f^*)^{-1}\mathbf{K}\Omega_f^*}$ *and assume that* $\mathbf{K}$ *is such that* $\hat{K} \in \mathcal{B}(\mathcal{H}^+(\mathbb{R}))$. *Given the quasi-affine map* $\Omega_f : \mathcal{H} \mapsto \mathcal{H}^+(\mathbb{R})$ *we define the map* $\Gamma(\Omega_f) : \mathcal{W} \mapsto \hat{\mathcal{W}}$ *through the following relations*

1. $\Gamma(\Omega_f)\mathbf{1}_{\Gamma_s(\mathcal{H})} = \mathbf{1}_{\Gamma_s(\mathcal{H}^+)}\Gamma(\Omega_f)$,
2. $\Gamma(\Omega_f)a(u) = \hat{a}(\Omega_f u)\Gamma(\Omega_f)$, $\forall u \in \mathcal{H}$,
3. $\Gamma(\Omega_f)a^\dagger(u) = \hat{a}^\dagger(\Omega_f u)\Gamma(\Omega_f)$, $\forall u \in \mathcal{H}$,
4. $\Gamma(\Omega_f)\lambda(\mathbf{K}) = \hat{\lambda}(\hat{K})\Gamma(\Omega_f)$.

□

Applying the transformation $\Gamma(\Omega_f)$ to an element $X \in \mathcal{W}$ we obtain an element $\hat{X} \in \hat{\mathcal{W}}$. We complete the definition of $\Gamma(\Omega_f)$ by determining its action on states defined on $\mathcal{W}$ and $\hat{\mathcal{W}}$:

Definition 6 ($\Gamma(\Omega_f)$, mapping of the state) *Let* $\Phi_{\mathcal{H}}$ *be the vacuum vector in* $\Gamma_s(\mathcal{H})$ *and let* $\Phi_{\mathcal{H}^+}$ *be the vacuum vector in* $\Gamma_s(\mathcal{H}^+(\mathbb{R}))$. *We define the action of* $\Gamma(\Omega_f)$ *on* $\Phi_{\mathcal{H}}$ *by*

$$\Gamma(\Omega_f)\Phi_{\mathcal{H}} = \Phi_{\mathcal{H}^+}\,.$$

□

As an example consider the application of $\Gamma(\Omega_f)$ to the exponential vector $e(u) \in \Gamma_s(\mathcal{H})$. For this vector we have a representation in the form $e(u) = \exp[a^\dagger(u)]\Phi_{\mathcal{H}}$. Hence

$$\Gamma(\Omega_f)e(u) = \Gamma(\Omega_f)e^{a^\dagger(u)}\Phi_{\mathcal{H}} = e^{\hat{a}^\dagger(\Omega_f u)}\Gamma(\Omega_f)\Phi_{\mathcal{H}} = e^{\hat{a}^\dagger(\Omega_f u)}\Phi_{\mathcal{H}^+} = e(\Omega_f u)\,.$$

Recall the definition of the basic creation, annihilation and conservation processes in $\Gamma_s(\mathcal{H})$

Definition 7 (creation, annihilation and conservation processes) *In the definition of* $\tilde{\mathcal{H}}$ *let* $\mathcal{H}_0$ *be chosen to be trivial i.e.,* $\mathcal{H}_0 = \mathbb{C}$ *so that* $\tilde{\mathcal{H}} \equiv \Gamma_s(\mathcal{H})$. *Let* m *be a* $\xi_{\mathcal{H}}$*-martingale and define* $(\xi, \mathbb{C}, \mathcal{H})$*-regular adapted processes* $A_m^\dagger := \{A_m^\dagger(t) \mid t \geq 0\}$ *and* $A_m := \{A_m(t) \mid t \geq 0\}$ *by*

1. $D(A_m^\dagger(t)) = D(A_m(t)) = \mathcal{E}(\mathcal{H})$,
2. $A_m^\dagger(t)e(u) = a^\dagger(m_t)e(u)$, $u \in \mathcal{H}$,
3. $A_m(t)e(u) = a(m_t)e(u)$, $u \in \mathcal{H}$.

The process $A_m^\dagger$ *is called the* creation process *and the process* A_m *is called the* annihilation process *in* $\Gamma_s(\mathcal{H})$. *Let* $\mathbf{K} \in \mathcal{B}(\mathcal{H})$ *be an operator such that* $[\mathbf{K}, \xi_{\mathcal{H}}([0,t])] = 0$ *for* $t \geq 0$ *and denote* $\mathbf{K}_t := \xi_{\mathcal{H}}([0,t])\mathbf{K}$. *Define the* $(\xi, \mathbb{C}, \mathcal{H})$*-regular adapted process* $\Lambda_{\mathbf{K}} = \{\Lambda_{\mathbf{K}}(t) \mid t \geq 0\}$ *in* $\Gamma_s(\mathcal{H})$ *by*

1. $D(\Lambda_{\mathbf{K}}(t)) = \mathcal{E}(\mathcal{H})$,
2. $\Lambda_{\mathbf{H}}(t)e(u) = \lambda(\mathbf{H}_t)e(u)$, $\forall u \in \mathcal{H}$.

The process $\Lambda_{\mathbf{K}}$ *is called the* conservation process *in* $\Gamma_s(\mathcal{H})$ *associated with* $\mathbf{K}$. □

The creation, annihilation and conservation processes in $\Gamma_s(\mathcal{H}^+(\mathbb{R}))$ are defined as in Definition 7 with obvious changes. We denote by $\hat{A}_{\hat{m}} = \{\hat{A}_{\hat{m}}(t) \mid t \geq 0\}$, $\hat{A}^\dagger_{\hat{m}} = \{\hat{A}^\dagger_{\hat{m}}(t) \mid t \geq 0\}$ and $\hat{\Lambda}_{\hat{K}} = \{\hat{\Lambda}_{\hat{K}(t)} \mid t \geq 0\}$ the creation, annihilation and conservation processes in $\Gamma_s(\mathcal{H}^+(\mathbb{R}))$ with $\hat{m}$ a $\xi_{\mathcal{H}^+}$-martingale and $\hat{K} \in \mathcal{B}(\mathcal{H}^+(\mathbb{R}))$. Applying the transformation $\Gamma(\Omega_f)$ to the creation and annihilation processes in Definition 7 we obtain

$$\Gamma(\Omega_f)A_m(t)e(u) = \quad \Gamma(\Omega_f)a(m_t)e(u) = \hat{a}(\hat{m}_t)\Gamma(\Omega_f)e(u) = \hat{A}_{\hat{m}}(t)\Gamma(\Omega_f)e(u)\,, \tag{45}$$

$$\Gamma(\Omega_f)A^\dagger_m(t)e(u) = \quad \Gamma(\Omega_f)a^\dagger(m_t)e(u) = \hat{a}^\dagger(\hat{m}_t)\Gamma(\Omega_f)e(u) = \hat{A}^\dagger_{\hat{m}}(t)\Gamma(\Omega_f)e(u)\,, \tag{46}$$

where according to Lemma 3 $\hat{m} = \Omega_f m$ is a $\xi_{\mathcal{H}^+}$-martingale. Eqns. (45), (46) can be written in short form

$$\Gamma(\Omega_f)A_m = \hat{A}_{\hat{m}}\Gamma(\Omega_f), \qquad \Gamma(\Omega_f)A^\dagger_m = \hat{A}^\dagger_{\hat{m}}\Gamma(\Omega_f)\,. \tag{47}$$

Let $\mathbf{K} \in \mathcal{B}(\mathcal{H})$ be such that $[\mathbf{K}, \xi_{\mathcal{H}}(E)] = 0$ for all $E \in \mathfrak{B}^+$ and define $\hat{K}$ according to Eq. (41). Then by Lemma 4 $\hat{K} \in \mathcal{B}(\mathcal{H}^+)$ and $[\hat{K}, \xi_{\mathcal{H}^+}(E)] = 0$. Moreover, denoting $\mathbf{K}_t = \xi_{\mathcal{H}}([0,t])\mathbf{K}$, the same lemma, and in particular Eq. (43), imply that

$$\widehat{(K_t)} = \overline{(\Omega_f^*)^{-1}\mathbf{K}_t\Omega_f^*} = \overline{(\Omega_f^*)^{-1}\xi_{\mathcal{H}}([0,t])\mathbf{K}\Omega_f^*} = \xi_{\mathcal{H}^+}([0,t])\hat{K} = \hat{K}_t\,.$$

Therefore, applying $\Gamma(\Omega_f)$ to the conservation process $\Lambda_{\mathbf{K}}$ and using Definition 5 we obtain

$$\begin{aligned}\Gamma(\Omega_f)\Lambda_{\mathbf{K}}(t)e(u) = \Gamma(\Omega_f)\lambda(\mathbf{K}_t)e(u) = \hat{\lambda}(\widehat{(K_t)})\Gamma(\Omega_f)e(u) = \\ = \hat{\lambda}(\hat{K}_t)\Gamma(\Omega_f)e(u) = \Lambda_{\hat{K}}(t)\Gamma(\Omega_f)e(u)\,.\end{aligned}$$

This equation can also be written in short form

$$\Gamma(\Omega_f)\Lambda_{\mathbf{K}} = \hat{\Lambda}_{\hat{K}}\Gamma(\Omega_f)\,. \tag{48}$$

Eq. (47) and Eq. (48) provide the transformation properties of the fundamental creation, annihilation and conservation processes under the mapping $\Gamma(\Omega_f)$. Since stochastic integration, and subsequently the construction of quantum stochastic differential equations, is defined with respect to these basic processes the transformation defined in Eq. (47) and Eq. (48) allows a mapping of stochastic processes defined with respect to $\tilde{\mathcal{H}}$ and the time observable $\mathbf{T}_F$ into stochastic processes defined with respect to $\tilde{\mathcal{H}}^+$ and the time observable $\hat{T}_F$. The procedure for doing this is demonstrated in the next example.

3.2 Mappings of quantum stochastic processes

We give an example of the mapping of stochastic processes induced by the map Ω_f through the transformation $\Gamma(\Omega_f)$. Before doing that we need to complete the discussion of the previous subsection with one more step. Suppose that $\mathcal{K}$ is a complex seperable Hilbert space and that ξ is an $\mathbb{R}^+$-valued time observable defined in $\mathcal{K}$. By this we mean that ξ is the spectral measure of some self-adjoint operator T with spectrum $\mathbb{R}^+$. Suppose that m and m' are two ξ-martingales. Then there is a complex measure $\ll m, m' \gg$ in $\mathbb{R}^+$ satisfying [25]

$$\ll m, m' \gg ([0,t]) = (m_t, m'_t)_{\mathcal{K}}, \qquad \forall t \geq 0\,.$$

For the example given below we shall need the transformation properties of this complex measure under the transformation $\Gamma(\Omega_f)$. Letting $\mathcal{K} = \mathcal{H}$ we recall that for every exponential vector in $\Gamma_s(\mathcal{H})$ (indeed this extends to every element of $\mathcal{E}(\mathcal{H})$)

$$[a(u), a^\dagger(v)]e(w) = (u, v)_{\mathcal{H}} e(w), \quad u, v, w \in \mathcal{H}\,.$$

Hence, if m, m' are two $\xi_{\mathcal{H}}$-martingales we have

$$[a(m_t), a^\dagger(m'_t)]e(u) = (m_t, m'_t)_{\mathcal{H}} e(u) = \ll m, m' \gg ([0,t])e(u)\,. \tag{49}$$

Applying the transformation $\Gamma(\Omega_f)$ to Eq. (49) we obtain

$$\begin{aligned}\Gamma(\Omega_f)\left[\ll m, m' \gg ([0,t])e(u)\right] &= \Gamma(\Omega_f)[a(m_t), a^\dagger(m'_t)]e(u) = [\hat{a}(\hat{m}_t), \hat{a}^\dagger(\hat{m}'_t)]\Gamma(\Omega_f)e(u) = \\ &= (\hat{m}_t, \hat{m}'_t)_{\mathcal{H}^+}\Gamma(\Omega_f)e(u) = \ll \hat{m}, \hat{m}' \gg ([0,t])\Gamma(\Omega_f)e(u)\,.\end{aligned}$$

We can now give an example of the mapping of stochastic processes in the form of the application of the transformation $\Gamma(\Omega_f)$ to an important class of quantum stochastic differential equations considered in [17, 25]. Let $\tilde{\mathcal{H}}$ and $\mathcal{H}_0$ be Hilbert spaces as in the left member of Eq. (44). Let O be a bounded operator in $\mathcal{H}_0$. We shall use the same notation for the extension $O = \{O(t) \mid t \geq 0\}$ of O to a constant regular adapted process with respect to the triplet $(\xi_{\mathcal{H}}, \mathcal{H}_0, \mathcal{H})$ defined by the ampliation $O(t) = O \otimes \mathbf{1}_{\Gamma_s(\mathcal{H})}$. In [17, 25] we find the following theorem which is stated here for the case of the Hilbert space $\mathcal{H}$ and the time observable $\mathbf{T}_F$:

Theorem 3 *Let $L, S, H \in \mathcal{B}(\mathcal{H}_0)$ and assume that S is unitary and H is self-adjoint. Let $\mathbf{P}$ be a projection in $\mathcal{H}$ commuting with the spectral measure $\xi_{\mathcal{H}}$ of the time observable $\mathbf{T}_F$. Let m be a $\xi_{\mathcal{H}}$-martingale such that $\mathbf{P}m_t = m_t$, $\forall t \geq 0$. Then there exists a unique unitary operator-valued $(\xi_{\mathcal{H}}, \mathcal{H}_0, \mathcal{H})$-regular adapted process $U = \{U(t) \mid t \geq 0\}$ satisfying*

$$dU = \left[L\, dA^\dagger_m + (S-1)d\Lambda_{\mathbf{P}} - L^*S\, dA_m - (iH + \frac{1}{2}L^*L)d \ll m, m \gg\right] U,\ U(0) = \mathbf{1}_{\tilde{\mathcal{H}}}\,. \tag{50}$$

Where $A^\dagger_m$, A_m and $\Lambda_{\mathbf{P}}$ are the fundamental creation, annihilation and conservation processes in $\Gamma_s(\mathcal{H})$. □

In order to be applicable to the stochastic process in Eq. (50) we first extend $\Gamma(\Omega_f)$ to a mapping $\tilde{\Gamma}(\Omega_f) : \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{W} \mapsto \mathcal{B}(\mathcal{H}_0) \otimes \hat{\mathcal{W}}$ via the relation

$$\tilde{\Gamma}(\Omega_f)(A \otimes B) = A \otimes \Gamma(\Omega_f)B, \qquad A \in \mathcal{B}(\mathcal{H}_0),\ B \in \mathcal{W}\,.$$

It is now possible to extend $\tilde{\Gamma}(\Omega_f)$ to a transformation of quantum stochastic process solutions of Eq. (50). First write a formal expression for the transformation of the stochastic process U into a stochastic process $\hat{U}$ by the (extended) mapping $\tilde{\Gamma}(\Omega_f)$ in the form $\tilde{\Gamma}(\Omega_f)U = \hat{U}\tilde{\Gamma}(\Omega_f)$. Then, in order to define the process $\hat{U}$ and thus complete the definition of the mapping $\tilde{\Gamma}(\Omega_f)$ apply the transformation $\tilde{\Gamma}(\Omega_f)$ to Eq. (50) and use the already known transformation properties of L, S, H, $\ll m, m \gg$ and A_m, $A^\dagger_m$, $\Lambda_{\mathbf{P}}$ to get

$$\begin{aligned}d\left[\tilde{\Gamma}(\Omega_f)U\right] &= \tilde{\Gamma}(\Omega_f)\left[L\, dA^\dagger_m + (S-1)d\Lambda_{\mathbf{P}} - L^*S\, dA_m - (iH + \frac{1}{2}L^*L)d \ll m, m \gg\right] U = \\ &= \left[L\, d\hat{A}^\dagger_{\hat{m}} + (S-1)d\hat{\Lambda}_{\hat{P}} - L^*S\, d\hat{A}_{\hat{m}} - (iH + \frac{1}{2}L^*L)d \ll \hat{m}, \hat{m} \gg\right] \tilde{\Gamma}(\Omega_f)U\,, \end{aligned} \tag{51}$$

where in Eq. (51) L, S and H stand for the constant stochastic processes $L(t) = L \otimes \mathbf{1}_{\Gamma_s(\mathcal{H}^+)}$, $S(t) = S \otimes \mathbf{1}_{\Gamma_s(\mathcal{H}^+)}$ and $H(t) = H \otimes \mathbf{1}_{\Gamma_s(\mathcal{H}^+)}$ in $\tilde{\mathcal{H}}^+$. Of course, Eq. (51) is still formal. However, the definition of $\hat{U}$ is clear, i.e., $\hat{U}$ is naturally defined as the solution of the quantum stochastic differential equation

$$d\hat{U} = \left[L\, d\hat{A}^\dagger_{\hat{m}} + (S-1)d\hat{\Lambda}_{\hat{P}} - L^* S\, d\hat{A}_{\hat{m}} - (iH + \frac{1}{2}L^* L)d \ll \hat{m}, \hat{m} \gg \right] \hat{U}, \quad \hat{U}(0) = \mathbf{1}_{\Gamma_s(\tilde{\mathcal{H}}^+)} \,. \tag{52}$$

The transition from Eq. (50) to Eq. (51) can then be written

$$\tilde{\Gamma}(\Omega_f) dU = \tilde{\Gamma}(\Omega_f) \left[L\, dA^\dagger_m + (S-1)d\Lambda_P - L^* S\, dA_m - (iH + \frac{1}{2}L^* L)d \ll m, m \gg \right] U$$

$$\Downarrow$$

$$d\hat{U}\, \tilde{\Gamma}(\Omega_f) = \left[L\, d\hat{A}^\dagger_{\hat{m}} + (S-1)d\hat{\Lambda}_{\hat{P}} - L^* S\, d\hat{A}_{\hat{m}} - (iH + \frac{1}{2}L^* L)d \ll \hat{m}, \hat{m} \gg \right] \hat{U}\, \tilde{\Gamma}(\Omega_f) \,.$$

The transformation $\tilde{\Gamma}(\Omega_f)$ constructed as above is well defined on solutions U of Eq. (50) and is given by the intertwining relation

$$\tilde{\Gamma}(\Omega_f) U = \hat{U} \tilde{\Gamma}(\Omega_f) \tag{53}$$

where $\hat{U}$ is a solution of Eq. (52). In particular, the transformation of the initial state $U(0)$ in Eq. (50) is given by

$$\tilde{\Gamma}(\Omega_f) U(0) = \tilde{\Gamma}(\Omega_f) \mathbf{1}_{\tilde{\mathcal{H}}} = \mathbf{1}_{\tilde{\mathcal{H}}^+} \tilde{\Gamma}(\Omega_f) = \hat{U}(0) \tilde{\Gamma}(\Omega_f)$$

where $\hat{U}(0) = \mathbf{1}_{\tilde{\mathcal{H}}^+}$ is the initial value for the process $\hat{U}$.

Eq. (53) is an analogue, for the class of quantum stochastic differential equations considered here, of the fundamental intertwining relation in Eq. (4). We observe that the stochastic process U is defined with respect to the spectral measure $\xi_{\mathcal{H}}$ of the time observable $\mathbf{T}_F$, the stochastic process $\hat{U}$ is defined with respect to the spectral measure $\xi_{\mathcal{H}^+}$ of the observable $\hat{T}_F$ and the transformation $\tilde{\Gamma}(\Omega_f)$ of stochastic processes is induced by the mapping $\Omega_f : \mathcal{H} \mapsto \mathcal{H}^+(\mathbb{R})$. It should be remarked at this point that following a procedure very similar to the one presented in this section it is possible to define an induced transformation $\tilde{\Gamma}(\Omega_f^*)$ mapping stochastic differential equations defined in $\tilde{\mathcal{H}}^+$ into stochastic differential equations defined in $\tilde{\mathcal{H}}$.

4 Summary

Time observables $\mathbf{T}_F$ and $\mathbf{T}_B$ for forward and backward quantum evolution were introduced in Section 2 above under the assumption that the quantum system under consideration is described by a complex seperable Hilbert space $\mathcal{H}$ and the generator of evolution is a self-adjoint Hamiltonian $\mathbf{H}$ on $\mathcal{H}$ satifying $\sigma(\mathbf{H}) = \sigma_{ac}(\mathbf{H}) = \mathbb{R}^+$. It was shown in Section 2 that $\mathbf{T}_F$ and $\mathbf{T}_B$ are positive, self-adjoint, semi-bounded operators in $\mathcal{H}$. The characterization of $\mathbf{T}_F$ as a forward time observable emerges from the fact, proved in Theorem 2 in Section 2, that if the quantum evolution is applied in the forward direction (i.e., for positive times) to

an initial state $g \in \mathcal{H}$ supported in a finite interval Δ in the spectrum of $\mathbf{T}_F$, then the evolved state $g(t) = \mathbf{U}(t)g = \exp[-i\mathbf{H}t]g$, $t \geq 0$ necessarily flows to higher parts of the spectrum of $\mathbf{T}_F$ as t increases. Indeed for any finite interval $\Delta \in \sigma(\mathbf{T}_F)$ the norm of the projection of $g(t)$ on the subspace of $\mathcal{H}$ corresponding to Δ by the spectral theorem (applied to $\mathbf{T}_F$) goes to zero as t goes to infinity. An analogous result holds for $\mathbf{T}_B$ for backward time evolution.

The basic mechanism enabling the definition of the time observable $\mathbf{T}_F$ involves a central ingredient of the semigroup decomposition formalism in the form of the fundamental intertwining relation appearing in Eq. (4). The fact that the characterization of $\mathbf{T}_F$ as a forward time observable is achieved through this intertwining relation, valid only for forward evolution, whereas the intertwining relation in Eq. (5), leading to the characterization of $\mathbf{T}_B$ as a backward time observable, is valid only for backward evolution, displays a built in time asymmetry in the theory. In Section 3 the discussion of this time asymmetry is opened up a bit further. it is shown there that, beyond its applicability to future directed Schrödinger type evolution, the operator $\mathbf{T}_F$ can, in fact, be used as a time observable in the construction of more general types of quantum processes clearly exhibiting future directed time evolution. Specifically, $\mathbf{T}_F$ and the corresponding Hardy space time observable $\hat{T}_F$ are used in the construction of quantum stochastic differential equations the solutions of which are (quantum) stochastic processes shown to satisfy an intertwining relation analogous to Eq. (4). Moreover, the map intertwining these quantum stochastic processes is, in fact, induced by the map Ω_f appearing in Eq. (4). Of course, the whole discussion can be repeated for backward time evolution using the operators $\mathbf{T}_B$ and $\hat{T}_B$.

Many questions are, of course, left open regarding the nature of the time observables and their appications. Here we mention briefly just a few. A first important question is whether the restriction on the spectrum of the Hamiltonian $\mathbf{H}$ put at the begining of the paper can be relaxed in such a way that meaningful time observables can still be defined. In addition, since $\mathbf{T}_F$ and $\mathbf{T}_B$ are operators in the physical space $\mathcal{H}$, one would like, if possible, to obtain expressions for them directly in terms of physical space variables without need for mappings to Hardy spaces. In this case what are the relations of the time ovservables to other observables of the physical system such as the Hamiltonian, momentum, position etc. ? In the context of irreversible quantum evolutions and the discusstion in Section 3 concerning quantum stochastic processes, one of the questions immediately arising is whether the intertwining relation in Eq. (53) can be shown to be more than just merely analogous to the one in Eq. (4) i.e., is it possible to show, for example, that Eq. (4) can be recovered in some sense from Eq. (53). These and related problems will be addressed elsewhere.

5 Appendix A

In this appendix we consider the properties of $\mathbf{T}_F^{-1}$ and $\hat{T}_F^{-1}$ as Hilbert space contractions on $\mathcal{H}$ and $\mathcal{H}^+(\mathbb{R})$ respectively. We recall that the *defect operators* for a contraction $T : \mathcal{H} \mapsto \mathcal{H}'$ and its adjoint $T^* : \mathcal{H}' \mapsto \mathcal{H}$ are given by

$$D_T := (1 - T^*T)^{1/2}, \qquad D_{T^*} := (1 - TT^*)^{1/2}$$

and the defect subspaces $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ are defined by

$$\mathcal{D}_T := \overline{D_T\mathcal{H}}, \qquad \mathcal{D}_{T^*} = \overline{D_{T^*}\mathcal{H}'}\,.$$

Hence for $\mathbf{T}_F$ and $\hat{T}_F^{-1}$

$$D_{\mathbf{T}_F^{-1}} = (1 - (\mathbf{T}_F^{-1})^2)^{1/2} = (1 - (\Omega_f^*\Omega_f)^2)^{1/2}, \qquad D_{\hat{T}_F^{-1}} = (1 - (\hat{T}_F^{-1})^2)^{1/2} = (1 - (\Omega_f\Omega_f^*)^2)^{1/2} \tag{54}$$

and for Ω_f and Ω_f^* we have

$$D_{\Omega_f} = (1 - \Omega_f^*\Omega_f)^{1/2} = (1 - \mathbf{T}_F^{-1})^{1/2}, \qquad D_{\Omega_f^*} = (1 - \Omega_f\Omega_f^*)^{1/2} = (1 - \hat{T}_F^{-1})^{1/2}\,. \tag{55}$$

We note that $(1 + \mathbf{T}_F^{-1})^{1/2}\mathcal{H} = \mathcal{H}$ and $(1 + \hat{T}_F^{-1})^{1/2}\mathcal{H}^+(\mathbb{R}) = \mathcal{H}^+(\mathbb{R})$. Using Eq. (54) and Eq. (55) we obtain

$$D_{\mathbf{T}_F^{-1}} = D_{\Omega_f}(1 + \mathbf{T}_F)^{1/2}, \quad D_{\hat{T}_F^{-1}} = D_{\Omega_f^*}(1 + \hat{T}_F)^{1/2}\,.$$

Hence we get that

$$\begin{aligned} \mathcal{D}_{\mathbf{T}_F^{-1}} &= \overline{D_{\mathbf{T}_F^{-1}}\mathcal{H}} = \overline{D_{\Omega_f}\mathcal{H}} = \mathcal{D}_{\Omega_f}, & (56)\\ \mathcal{D}_{\hat{T}_F^{-1}} &= \overline{D_{\hat{T}_F^{-1}}\mathcal{H}^+(\mathbb{R})} = \overline{D_{\Omega_f^*}\mathcal{H}^+(\mathbb{R})} = \mathcal{D}_{\Omega_f^*}\,. & (57) \end{aligned}$$

In addition we have the relations

$$\mathcal{D}_{T^*} = \overline{T\mathcal{D}_T} \oplus \mathrm{Ker}\,(T^*), \qquad \mathcal{D}_T = \overline{T^*\mathcal{D}_{T^*}} \oplus \mathrm{Ker}\,(T)\,.$$

so that by the fact that Ω_f and Ω_f^* are both injective we have

$$\mathcal{D}_{\Omega_f^*} = \overline{\Omega_f\mathcal{D}_{\Omega_f}}, \qquad \mathcal{D}_{\Omega_f} = \overline{\Omega_f^*\mathcal{D}_{\Omega_f^*}}\,.$$

Hence

$$\Omega_f\mathcal{D}_{\Omega_f} \subset \mathcal{D}_{\Omega_f^*}, \qquad \Omega_f^*\mathcal{D}_{\Omega_f^*} \subset \mathcal{D}_{\Omega_f}\,.$$

Eq. (56) then implies that

$$\Omega_f\mathcal{D}_{\mathbf{T}_F^{-1}} \subset \mathcal{D}_{\hat{T}_F^{-1}}, \qquad \Omega_f^*\mathcal{D}_{\hat{T}_F^{-1}} \subset \mathcal{D}_{\mathbf{T}_F^{-1}}\,. \tag{58}$$

In fact, in Eq. (58) the left hand members are dense in the right hand members. The *charateristic function* $\Theta_{\mathbf{T}_F^{-1}} : \mathcal{D}_{\mathbf{T}_F^{-1}} \mapsto \mathcal{D}_{\mathbf{T}_F^{-1}}$ of T_F^{-1} is given by

$$\Theta_{\mathbf{T}_F^{-1}}(\lambda) = \left(-\mathbf{T}_F^{-1} + \lambda D_{\mathbf{T}_F^{-1}}(1 - \lambda\mathbf{T}_F^{-1})^{-1}D_{T_F^{-1}}\right)\big|\mathcal{D}_{\mathbf{T}_F^{-1}}\,. \tag{59}$$

Hence, with the help of Eq. (12), we can write

$$\begin{aligned} \Theta_{\mathbf{T}_F^{-1}}(\lambda) &= \left[\lambda(1 - \lambda\Omega_f^*\Omega_f)^{-1} - \left(\Omega_f^*\Omega_f + \lambda(\Omega_f^*\Omega_f)^2(1 - \lambda\Omega_f^*\Omega_f)^{-1}\right)\right]\Big|\mathcal{D}_{\mathbf{T}_F^{-1}} = \\ &= \left[\lambda(1 - \lambda\Omega_f^*\Omega_f)^{-1} - \Omega_f^*\left(1 + \lambda\Omega_f(1 - \lambda\Omega_f^*\Omega_f)^{-1}\Omega_f^*\right)\Omega_f\right]\Big|\mathcal{D}_{\mathbf{T}_F^{-1}} = \\ &= \left[R_{\mathbf{T}_F^{-1}}(\lambda^{-1}) - \lambda^{-1}\Omega_f^*R_{\hat{T}_F^{-1}}(\lambda^{-1})\Omega_f\right]\Big|\mathcal{D}_{\mathbf{T}_F^{-1}}\,. \end{aligned} \tag{60}$$

Exchanging Ω_f and Ω_f^* in Eq. (60) we get the characteristic function of $\hat{T}_F^{-1}$

$$\Theta_{\hat{T}_F^{-1}}(\lambda) = \left[R_{\hat{T}_F^{-1}}(\lambda^{-1}) - \lambda^{-1}\Omega_f R_{\mathbf{T}_F^{-1}}(\lambda^{-1})\Omega_f^*\right]\Big|\mathcal{D}_{\hat{T}_F^{-1}}\,. \tag{61}$$

Multiplying Eq. (61) from the left by Ω_f^* and takeing notice of the fact that the identity $\Omega_f \mathbf{T}_F^{-1} = \hat{T}_F^{-1}\Omega_f^*$ imply that $\Omega_f R_{\mathbf{T}_F^{-1}}(z) = R_{\hat{T}_F^{-1}}(z)\Omega_f$ and $\Omega_f^* R_{\hat{T}_F^{-1}}(z) = R_{\mathbf{T}_F^{-1}}(z)\Omega_f^*$ we have

$$\Omega_f^*\Theta_{\hat{T}_F^{-1}}(\lambda) = \left[R_{\mathbf{T}_F^{-1}}(\lambda^{-1}) - \lambda^{-1}\Omega_f^* R_{\hat{T}_F^{-1}}(\lambda^{-1})\Omega_f\right]\Omega_f^*\Big|\mathcal{D}_{\hat{T}_F^{-1}}\,. \tag{62}$$

We now compare Eq. (62) and Eq. (60) and take into account Eq. (58) in order to obtain

$$\Omega_f^*\Theta_{\hat{T}_F^{-1}}(\lambda) = \Theta_{\mathbf{T}_F^{-1}}(\lambda)\Omega_f^*\big|\mathcal{D}_{\hat{T}_F^{-1}}\,.$$

Following a similar procedure we find also that

$$\Omega_f\Theta_{\mathbf{T}_f^{-1}}(\lambda) = \Theta_{\hat{T}_F^{-1}}(\lambda)\Omega_f\big|\mathcal{D}_{\mathbf{T}_F^{-1}}\,.$$

Thus we arrived at the following proposition

Proposition 4 *Let $\Theta_{\mathbf{T}_F^{-1}}(z)$ and $\Theta_{\hat{T}_F^{-1}}(z)$ be the characteristic and $\mathcal{D}_{\mathbf{T}_F^{-1}}$, $\mathcal{D}_{\hat{T}_F^{-1}}$ be the defect subspaces of $\mathbf{T}_F^{-1}$ and $\hat{T}_F^{-1}$ respectively. Then we have*

$$\Omega_f^*\Theta_{\hat{T}_F^{-1}}(z) = \Theta_{T_F^{-1}}(z)\Omega_f^*\big|\mathcal{D}_{\hat{T}_F^{-1}}\,.$$

$$\Omega_f\Theta_{\mathbf{T}_f^{-1}}(\lambda) = \Theta_{\hat{T}_F^{-1}}(\lambda)\Omega_f\big|\mathcal{D}_{\mathbf{T}_F^{-1}}\,.$$

□

The equality of spectrums is an immediate corollary of the above proposition:

Corollary 1 *The spectrums of $\mathbf{T}_F^{-1}$ and $\hat{T}_F^{-1}$ satisfy $\sigma(\mathbf{T}_F^{-1}) = \sigma(\hat{T}_F^{-1})$.* □

Proof:

We can use either of the two equations in Proposition 4. Using the first equation the corollary immediately follows from the fact that $\mathrm{Ker}\,(\Omega_f^*) = \{0\}$, the fact that $\Omega_f^*\mathcal{D}_{\hat{T}_F^{-1}}$ is dense in $\mathcal{D}_{\mathbf{T}_F^{-1}}$ and from a theorem of Sz.-Nagy and C. Foias (see [32], Chap. VI, Sec. 4) stating that the characteristic function of a contraction T determines uniquely the spectrum of T. ■

Acknowledgments

I wish to thank Prof. L.P. Horwitz for useful discussions during the preparation of this paper. Research supported by ISF under Grant No. 1282/05 and by the Center for Avanced Studies in Mathematics at Ben-Gurion University.

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